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Contents

Discrete time

Historically, financial mathematics originated in continuous time - like in the famous works from Black & Scholes (1973) and **?** and the main driving force was a Brownian motion. It seems very plausible, that a large number of traders who act independently can be approximated by a Gaussian distribution through the central limit theorem, such that this is a very appealing setup.

However, this requires the full power of stochastic integration and the technical details are quite subtle. It is remarkable, that the main concepts of financial markets, like absence of arbitrage, the first and second fundamental theorem can be proven in discrete time without the need to dive into the technicalities while providing similarly deep insights. I therefore believe, it is a good start to spend some time on discrete time.

A discrete-time financial market

An excellent introduction to the field is Föllmer & Schied (2016). We follow this book for the introduction and directly start in a multi-period financial market. The advantage of this approach - as we will soon see - is that a multi-period market essentially can be reduced to a one-period market.

To this end we fix a probability space $(\Omega, \mathscr{F}, P)$. A financial market consists of one primary risk-free asset S^0 which is assumed to be strictly positive. Furthermore, we have d risky assets $S = (S^1, \dots, S^d)$ which are assumed to be non-negative. All assets are described as stochastic processes on the time interval $\mathbb{T} = \{0, \dots, T\}$.

The information flow is described by the filtration $\mathbb{F} = (\mathscr{F}_t)_{t \in \mathbb{T}}$. We denote by $\bar{S} = (S^0, S)$ the $d+1$-dimensional stochastic process including the risk-free account. We assume that $\bar{S}$ is adapted to the filtration $\mathbb{F}$.

Definition 1. A *trading strategy* $\bar{H}$ is a predictable, $d+1$-dimensional stochastic process. The trading strategy is *self-financing*, if

$$\bar{H}_t \cdot \bar{S}_t = \bar{H}_{t+1} \cdot \bar{S}_t, \qquad t = 1, \dots, T-1.$$

Intuitively, a self-financing trading strategy does not require external funds while rebalancing at time t, neither does it produce a consumable profit.

Let us denote the increments of S by

$$\Delta S_t = S_t - S_{t-1}, \qquad t = 1, \dots, T.$$

It is a remarkable result, that for a self-financing trading strategy, the position at time t can be decomposed in the initial value plus the gains from trading. The gains itself can be written as a (discrete) stochastic integral.

Lemma 2. *For a self-financing trading strategy $\bar{H}$, we obtain that*

$$\bar{H}_t \cdot \bar{S}_t = \bar{H}_1 \cdot \bar{S}_0 + \sum_{k=1}^{t} \bar{H}_k \cdot \Delta \bar{S}_k, \qquad 1 \leq t \leq T.$$

Proof. This follows in two steps:

$$\begin{aligned}
\bar{H}_t \cdot \bar{S}_t &= \bar{H}_t \cdot \bar{S}_t + \bar{H}_{t-1} \cdot \bar{S}_{t-1} - \bar{H}_{t-1} \cdot \bar{S}_{t-1} \\
&= \bar{H}_t \cdot \bar{S}_t + \bar{H}_{t-1} \cdot \bar{S}_{t-1} - \bar{H}_t \cdot \bar{S}_{t-1} \\
&= \bar{H}_t \cdot (\bar{S}_t - \bar{S}_{t-1}) + \bar{H}_{t-1} \cdot \bar{S}_{t-1} \\
&= \sum_{k=2}^{t} \bar{H}_k \cdot (\bar{S}_k - \bar{S}_{k-1}) + \bar{H}_1 \cdot \bar{S}_1
\end{aligned}$$

where we used that $\bar{H}$ is self-financing. For the last time step we obtain,

$$\begin{aligned}
\bar{H}_1 \cdot \bar{S}_1 &= \bar{H}_1 \bar{S}_1 + \bar{H}_1 \cdot \bar{S}_0 - \bar{H}_1 \cdot \bar{S}_0 \\
&= \bar{H}_1 (\bar{S}_1 - \bar{S}_0) + \bar{H}_1 \cdot \bar{S}_0
\end{aligned}$$

and the claim follows. □

Example 3 (Bank account). A typical example for S^0 is the bank account. The bank account starts at $S_0^0 = 1$ and offers the interest rate r_t from $t-1$ to tt. Note that r_t is of course already known at $t-1$ and hence predictable. Hence,

$$S_t^0 = \prod_{s=1}^{t} (1 + r_s).$$

We always require $r_t > -1$. But often one additionally assumes that $r_t \geq 0$.

Moving to discounted quantities

An important step - economically, and mathematically - is to move to discounted quantities. While this simplifies that setup drastically, it also has a number of subtle consequences (in particular in continuous time).

We introduce the *discounted price process*

$$X_t^i := \frac{S_t^i}{S_t^0}, \quad t = 0, \ldots, T, \quad i = 0, \ldots, d.$$

Note that $X^0 \equiv 1$ and in particular $\Delta X_t^0 = 0$. As previously, we use the notation $\bar{H} = (H^0, H)$.

Definition 4. The *discounted value process* $V = V^{\bar{H}}$ of the trading strategy $\bar{H}$ is given by

$$V_t := \bar{H}_t \cdot \bar{X}_t, \quad t = 1, \dots, T$$

with $V_0 := \bar{H}_1 \cdot \bar{X}_0$. The *discounted gains process* $G = G^{\bar{H}}$ is

$$G_t := \sum_{k=1}^{t} H_k \cdot \Delta X_k, \quad t = 1, \dots, T$$

with $G_0 = 0$.

Of course,

$$G_t = \sum_{k=1}^{t} \bar{H}_k \cdot \Delta \bar{X}_k,$$

which explains why we can switch from $\bar{H}$ to H on discounted quantities.

Proposition 5. *Consider the trading strategy $\bar{H}$. Then the following are equivalent:*

(i) $\bar{H}$ *is self-financing,*

(ii) $\bar{H}_t \cdot \bar{X}_t = \bar{H}_{t+1} \cdot \bar{X}_t, \quad t = 1, \dots, T-1,$

(iii) $V_t = V_0 + G_t$ *for* $0 \leqslant t \leqslant T$.

Proof. By definition, self-financing is equivalent to

$$\begin{aligned} & \bar{H}_t \cdot \bar{S}_t = \bar{H}_{t+1} \cdot \bar{S}_t & & t = 0, \dots, T-1, \\ \Leftrightarrow \quad & \bar{H}_t \cdot \frac{\bar{S}_t}{S_t^0} = \bar{H}_{t+1} \cdot \frac{\bar{S}_t}{S_t^0}, & & t = 0, \dots, T-1, \end{aligned}$$

since $S^0 > 0$. This yields equivalence of (i) and (ii). For the next step we compute the increments of the value process. By (ii),

$$V_t - V_{t-1} = \bar{H}_{t+1} \cdot \bar{X}_{t+1} - \bar{H}_t \cdot \bar{X}_t = \bar{H}_{t+1} \cdot (\bar{X}_{t+1} - \bar{X}_t) = H_{t+1} \cdot (X_{t+1} - X_t).$$

Hence,

$$V_t - V_0 = \sum_{s=1}^{t} H_s \cdot (X_s - X_{s-1}), \quad t = 1, \dots, T$$

and the conclusion follows. □

Remark 6 (Trading strategies)**.** If we start with a d-dimensional trading strategy H, we can determine the associated self-financing strategy $\bar{H}$ as follows: choose H^0 according to

$$H_{t+1}^0 - H_t^0 = -(H_{t+1} - H_t) \cdot X_t, \qquad t = 0, \dots, T-1,$$

and $H_1^0 = V_0 - H_1 \cdot X_0$. In the following, if we speak of a self-financing trading strategy H, we mean equivalently this associated strategy $\bar{H}$.

Arbitrage and martingales

The central concept for our analysis of financial markets is the concepts of arbitrage.

Definition 7. An *arbitrage* is a self-financing trading strategy H, such that the associated discounted value process V satisfies

(i) $V_0 \leqslant 0$,

(ii) $V_T \geqslant 0$, and

(iii) $P(V_T > 0) > 0$.

If there are no arbitrages on a financial market, we call it arbitrage-free.

If we recall our introduction, we realize that a financial market consists of the triplet $(\mathbb{F}, S, P)$. It can be easily shown that the above conditions are equivalent to the same conditions for the undiscounted value process $(\bar{H}_t \cdot \bar{S}_t)$.

Proposition 8. *A financial market is free of arbitrage if and only if every one-period financial market* (S_t, S_{t+1})*,* $t = 0, \dots, T-1$ *is free of arbitrage.*

Proof. The idea of the proof is to show the equivalence of the negotiations: there exists an arbitrage if and only if for a $t \in \{1, \dots, T\}$ there exists a $\mathscr{F}_{t-1}$-measurable random $\xi \in \mathbb{R}^d$, such that $\xi \cdot \Delta X_t \geq 0$ P-a.s. and $P(\xi \cdot \Delta X_t > 0) > 0$.

For the first part we start with an arbitrage and show that there exists a single period with an arbitrage: let $\bar{H}$ be an arbitrage with value process V. Let

$$t := \min \big\{ s \in \{1, \dots, T\} : V_s \geqslant 0 \text{ und } P(V_s > 0) > 0 \big\}$$

be a deterministic time with the convention that $\min \varnothing = \infty$. Since $\bar{H}$ is an arbitrage, we obtain that $t \leqslant T$. There are two possibilities to be taken into account: either $V_{t-1} = 0$, or $P(V_{t-1} < 0) > 0$. In the first case, we are ready, since

$$H_t \cdot (X_t - X_{t-1}) = V_t - V_{t-1} = V_t,$$

so $\xi = H_t$ does the job.

For the second case we choose $\xi := H_t \mathbb{1}_{\{V_{t-1} < 0\}}$. Then ξ is $\mathscr{F}_{t-1}$-measurable and

$$\xi \cdot (X_t - X_{t-1}) = (V_t - V_{t-1}) \mathbb{1}_{\{V_{t-1} < 0\}} \geqslant -V_{t-1} \mathbb{1}_{\{V_{t-1} < 0\}} \geqslant 0.$$

Now observe that the r.h.s. is positive with positive probability and the first part is finished.

The other direction is straightforward: set $H_s = \xi \mathbb{1}_{\{s=t\}}$ and construct the associated self-financing trading strategy $\bar{H}$, which is an arbitrage. □

Remark 9. It is interesting to see that the property to deduce absence of arbitrage from one time period only breaks down if one allows for two filtrations, see Kabanov & Stricker (2006). A fundamental theorem for two markets with two filtrations is still an open research question.

Definition 10. A probability measure Q on $(\Omega, \mathscr{F}_T)$ is called *martingale measure*, if X is a Q-martingale.

We note that a martingale measure refers to the *discounted* price process being a martingale. It is a bit surprising that an artificial probability measure takes up such a prominent role in no-arbitrage theory. We will see later, why. Recall that Q is called absolutely continuous with respect to P ($Q \ll P$) if $P(F) = 0$ for an $F \in \mathscr{F}_T$ implies that $Q(F) = 0$. Q is called equivalent to P ($P \sim Q$), if $Q \ll P$ and $P \ll Q$.

Martingale measures

We denote the set of *equivalent martingale measures* by $\mathscr{M}_e(\mathbb{F}) = \mathscr{M}_e$. If we assume that the initial filtration is trivial, the following result can already be obtained. It is remarkable that the integrability condition of a martingale can be obtained from no-arbitrage (equivalently the existence of a martingale measure as we will see later) and a substantially weakened integrability.

Satz 11. *Let $\mathscr{F}_0 = \{\varnothing, \Omega\}$. Then, the following are equivalent:*

(i) $Q \in \mathscr{M}_E(\mathbb{F})$

(ii) For any bounded self-financing trading strategy $\bar{H}$, $V^{\bar{H}}$ is a Q-martingale.

(iii) For any self-financing trading strategy $\bar{H}$ such that $E_Q[(V_T^{\bar{H}})^-] < \infty$ $V^{\bar{H}}$ is a Q- martingale.

(iv) For any self-financing trading strategy $\bar{H}$ with $V_T = V_T^{\bar{H}} \geqslant 0$ it holds that

$$V_0 = E_Q[V_T].$$

Proof. i) $\Rightarrow$ ii): we start with a bounded self-financing strategy $\bar{H}$, i.e. $|H_t^i| \leqslant c$, for $i = 0, \ldots, d$, $t = 0, \ldots, T$. Then, integrability of $V = V^{\bar{H}}$ follows from the integrability of X since

$$|V_t| \leqslant |V_0| + \sum_{k=1}^{t} c \cdot \sum_{i=1}^{d} \Big(|X_k^i| + |X_{k-1}^i| \Big).$$

Moreover, we have that

$$\begin{aligned} E_Q[V_t|\mathscr{F}_{t-1}] &= E_Q[V_{t-1} + \bar{H}_t \cdot (\bar{X}_t - \bar{X}_{t-1})|\mathscr{F}_{t-1}] \\ &= V_{t-1} + \bar{H}_t \cdot (E_Q[\bar{X}_t|\mathscr{F}_{t-1}] - \bar{X}_{t-1}) = V_{t-1}. \end{aligned}$$

ii) $\Rightarrow$ iii): We want to show the martingale property under a minimal integrability assumption. We start with the assumption that $E_Q[V_T^-] < \infty$. Then $E_Q[V_T]$ and, similarly, $E_Q[V_T|\mathscr{F}_{T-1}]$ is well-defined (though possibly not finite).

Consider $a > 0$. Then,

$$\begin{aligned} E_Q[V_T|\mathscr{F}_{T-1}]\mathbb{1}_{\{|\bar{H}_T| \leqslant a\}} &= E_Q[\bar{H}_T \cdot \bar{X}_T \mathbb{1}_{\{|\bar{H}_T| \leqslant a\}}|\mathscr{F}_{T-1}] \\ &= E_Q[\bar{H}_T(\bar{X}_T - \bar{X}_{T-1})\mathbb{1}_{\{|\bar{H}_T| \leqslant a\}}|\mathscr{F}_{T-1}] + V_{T-1}\mathbb{1}_{\{|\bar{H}_T| \leqslant a\}} \\ &= V_{T-1}\mathbb{1}_{\{|\bar{H}_T| \leqslant a\}}. \end{aligned} \tag{12}$$

Now with $a \to \infty$, $\{|\bar{H}_T| \leqslant a\} \to \Omega$, since $\bar{H}_T$ is a $\mathbb{R}^{d+1}$-value random variable. Hence, $E_Q[V_T|\mathscr{F}_{T-1}] = V_{T-1}$ Q-a.s. The next step is to show $E_Q[V_{T-1}^-] < \infty$ with Jensens' inequality:

this follows since

$$E_Q[V_{T-1}^-] = E_Q[E_Q[V_T|\mathscr{F}_{T-1}]^-] \leqslant E_Q[E_Q[V_T^-|\mathscr{F}_{T-1}]] = E_Q[V_T^-] < \infty.$$

Proceeding iteratively we obtain $E_Q[V_t^-] < \infty$ as well as $E_Q[V_t|\ \mathscr{F}_{t-1}] = V_{t-1}$.

Then,

$$V_0 = E_Q[V_t].$$

Since $E_Q[V_t^-]$, the expectation is well-defined. Moreover, $V_0 \in \mathbb{R}$, so $E_Q[V_t^+] < \infty$, and hence $E_Q[|V_t|] < \infty$. We obtain integrability and hence V is a Q-martingale.

iii) $\Rightarrow$ iv): clear.

iv) $\Rightarrow$ i): First, we show integrability of X_t^i. This can be achieved using $X_t^i = X_0^i + \sum_{s=1}^t \Delta X_s^i$. We choose accordingly $H_s^j = \mathbb{1}_{\{s\leq t\}}\mathbb{1}_{\{j=i\}}$, such that $G_T = \sum_{s=1}^t \Delta X_s^i$ and, (by Proposition (iii)), $V_0 = X_0^i$. Moreover, $V_T = X_t^i \geq 0$. Then we can apply (iv), such that

$$\infty > X_0^i = V_0 = E_Q[V_T] = E_Q[X_t^i] = E_Q[|X_t^i|]. \tag{13}$$

The next goal is to show

$$E_Q[X_{t+1}^i 1_F] = E_Q[X_t^i 1_F] \qquad \forall F \in \mathscr{F}_t$$

and for $1 \leq t \leq T$. For this, we aim at a similar strategy, but $H_s = \mathbb{1}_{\{s\leq t\}}\mathbb{1}_F - \mathbb{1}_{\{s\leq t-1\}}\mathbb{1}_F$ is not possible since $\mathbb{1}_F$ is only $\mathscr{F}_t$-measurable. Instead we search for a strategy which achieves $V_T = X_t^i\mathbb{1}_F + X_{t+1}^i\mathbb{1}_{F^C} \geq 0$. Note that $V_T = X_t^i + \Delta X_{t+1}^i\mathbb{1}_{F^C}$ which can be achieved by the trading strategy

$$H_s^i = \mathbb{1}_{\{s\leq t\}} + \mathbb{1}_{F^C} 1_{\{s=t+1\}},$$

and $H^j = 0$ for $j \neq i$ together with $V_0 = X_0^i$, as above. Again, by (iv),

$$X_0^i = V_0 = E_Q[V_T] = E_Q[\mathbb{1}_F X_t^i + \mathbb{1}_{F^C} X_{t+1}^i].$$

Together with (13),

$$E_Q[X_{t+1}^i] = X_0^i = E_Q[\mathbb{1}_F X_t^i + \mathbb{1}_{F^C} X_{t+1}^i],$$

hence $E_Q[1_F X_t^i] = E_Q[1_F X_{t+1}^i]$, and the claim follows. □

The fundamental theorem

The following fundamental theorem relates absence of arbitrage with a very simple criterion - the existence of a martingale measure. This measure can be used for pricing in a very simple manner, which explains the immense success of this approach.

Theorem 14 (FTAP). *A financial market is free of arbitrage, if and only if $\mathscr{M}_e(\mathbb{F}) \neq \varnothing$. In this case there exists a $Q \in \mathscr{M}_e$ with bounded density dQ/dP.*

This theorem is proved in several steps. The first step, which is the most important step in applications, is surprisingly easy.

Proposition 15. *If $\mathscr{M}_e \neq \varnothing$, then there is no arbitrage.*

Proof. We use proposition 11 (iii). Assume that H is an arbitrage with discounted value process V and chose a $Q \in \mathscr{M}_e$.

Note that with $V_0 \leqslant 0$ P-a.s. it holds that $V_0 \leqslant 0$ Q-a.s. In the same manner, we obtain that $V_T \geq 0$ Q-a.s. and hence $E_Q[V_T^-] = 0 < \infty$.

Since H is an arbitrage, $P(V_T > 0) > 0$, and hence $Q(V_T > 0) > 0$. This implies $E_Q[V_T] > 0$. Since Proposition 11 (iii) yields that V is a Q-martingale, i.e.

$$V_0 = E_Q[V_T] > 0,$$

we obtain a contradiction to $V_0 \leqslant 0$. $\square$

Examples

Let us visit shortly some examples which illustrate the importance to the application of Proposition 15. Note that if we add a Q-martingale as additional coordinate to the price process X the market remains arbitrage-free. It is therefore natural to use the *risk-neutral pricing rule* for pricing additional contingent claims.

Consider n $\mathscr{F}_T$-measurable contingent claim with (discounted) payoff $C_T \geq 0$ and let

$$X_t^{d+1} = E_Q[C_T | \mathscr{F}_t], \qquad t \in \mathbb{T}.$$

Then, the extended market $\widetilde{X} = (X^1, \dots, X^{d+1})$ is free of arbitrage.

Example 16 (Black-Scholes Model)**.** The famous Black-Scholes modelgives the stock price under P as a geometric Brownian motion, precisely:

$$dS_t = S_t \mu dt + S_t \sigma dW_t$$

with a Brownian motion W and initial value $\mathbb{R} \ni S_0 > 0$. The unique strong solution of this SDE is given by

$$S_t = S_0 \exp\Big((\mu - \sigma^2/2)t + \sigma W_t \Big), \qquad t \geq 0.$$

A typical derivative is an *European call* which offers the pay-off

$$(S_T - K)^+.$$

If we additionally assume that the Bank account is e^{rt}, then the Girsanov theorem shows that the measure $dQ = L_T dP$ with

$$dL_t = -L_t \lambda dW_t,$$

and $\lambda = {}^{r-\mu}/_{\sigma}$ is a martingale measure. Under Q, $\widetilde{W}_t = W_t + \lambda t,\ t \geq 0$ is a Brownian motion and so

$$dS_t = S_t r dt + S_t \sigma d\widetilde{W}_t,$$

and – of course – $dX_t = S_t \sigma d\widetilde{W}_t$. Let us compute shortly the price of the call option,

$$E_Q[C_T] = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2), \tag{17}$$

$$d_{1/2} = \frac{\log \frac{S_0}{Ke^{-rT}} \pm \frac{\sigma^2 T}{2}}{\sigma \sqrt{T}} \tag{18}$$

While this model is in continuous time, the discrete analogue $S = (S_{t_i})_{i=0,\dots,n}$ with $t_i = \delta i$ is of course in discrete time. It is remarkable that the continuous time model has only one equivalent martingale measure, while the discrete time model has many.

Example 19 (The binomial model)**.** An example in discrete time which is very illustrative, is the Binomial model (also called the Cox-Ross-Rubinstein model). Here, we assume that

$$S_t = S_0 \prod_{i=1}^{t} \xi_i, \qquad t = 1, \dots, T$$

with $\xi_i \in \{1+u, 1+d\}$, $-1 \le d \le u$. The bank account is $S_t^0 = (1+r)^t$, $r > -1$. This model has no-arbitrage if and only if

$$d < r < u.$$

Indeed, for a martingale measure we need

$$1 = E_Q[(1+r)^{-1} \cdot \xi_t | \mathscr{F}_{t-1}] = (1+r)^{-1} \Big(q_t \cdot (1+u) + (1-q_t) \cdot (1+d) \Big)$$

which is equivalent to

$$q_t = \frac{r-d}{u-d}.$$

It is remarkable to realise that under Q, (ξ_t) are i.i.d., while this assumption is of course not necessary under P. We also may see that the martingale measure is actually used to characterise the convex hull of $(1+d, 1+u)$ and so a geometric interpretation of existence of a martingale measure is the property that the return of the bank account $1+r$ lies in the interior of the convex span of the return of the stock.

It is also interesting to see that for a contingent claim with price C_{t-1} at $t-1$ and values $\{C_t^+, C_t^-\}$ at t, the replicating strategy is determined by

$$H_t = \frac{C_t^+ - C_t^-}{S_{t-1} \cdot (u-d)}.$$

Note that this strategy is unique !

Complete markets

We call a contingent claim C_T *replicable*, if there exists a self-financing trading strategy, such that $V_T = C_T$. We call an arbitrage-free market *complete*, if every contingent claim is replicable.

Proposition 20. *Assume that $\mathscr{F}_0 = \{\emptyset, \mathscr{F}\}$. Then, an arbitrage-free market is complete, if and only if*

$$\mathscr{M}_e = \{Q\}.$$

Proof. Assume that the market is complete. We first note that a replicable claim has a unique price: we first note that by Proposition 11 the value processes of replicable trading strategies are martingales, since $V_T = C_T \ge 0$. Hence, $E_Q[V_T] = V_0 < \infty$ for all $Q \in \mathscr{M}_e$. This implies that the price of a *replicable* contingent claim C_T is unique, since for any replicable trading strategy

$$V_0 = E_Q[V_T] = E_Q[C_T]$$

and the right hand side does not depend on V_T.

Now we show that for $Q, Q' \in \mathscr{M}_e$ it holds that $Q = Q'$. Indeed, consider $C_T = \mathbb{1}_F$ for any $F \in \mathscr{F}_T$. Then

$$Q(F) = E_Q[\mathbb{1}_F] = E_{Q'}[\mathbb{1}_F] = Q'(F).$$

We postpone the second assertion - since it needs a little bit more technique. ☐

The second part of the proof of the FTAP

Now we return to the fundamental theorem. The main goal is to show the existence of an equivalent martingale measure. Since this relates to linear functionals, our proof will reside on the Hahn-Banach theorem

Since we already know that it is sufficient to consider one period only, we study the gains which can be obtained in a self-financing manner with zero initial investment. Fix $0 < t \leq T$. Then these gains are given by

$$K = \{H \cdot \Delta X_t : \ H \in L^0(P, \mathscr{F}_{t-1}, \mathbb{R}^d)\}.$$

Then, absence of arbitrage (NA) is equivalent to

$$K \cap L^0_+(\mathscr{F}_t, P, \mathbb{R}) = \{0\}.$$

Since, t is fixed we shortly write $L^0_+(\mathscr{F}_t, P, \mathbb{R}) = L^0_+$. One central and very useful observation in the following theorem is that the set K can be replaced by the claims which can be *super-replicated* which is given by the set $K - L^0_+$.

Satz 21. *Consider the one period-market from $t-1$ to t. Then, the following are equivalent:*

(i) $K \cap L^0_+ = \{0\}$,

(ii) $(K - L^0_+) \cap L^0_+ = \{0\}$,

(iii) there exists an equivalent martingale measure with bounded density,

(iv) there exists an equivalent martingale measure.

Proof. We show (iv) $\Rightarrow$ (i) $\Leftrightarrow$ (ii) and (iii) $\Rightarrow$ (iv). The part (ii) $\Rightarrow$ (iii) is the most difficult part and will be treated separately.

(iv) $\Rightarrow$ (i): we aim at a contradiction. Consider $Q \in \mathscr{M}_e$ and assume there exists $H \in L^0(P, \mathscr{F}_{t-1}, \mathbb{R}^d)$ such that $H \cdot (X_t - X_{t-1}) \geqslant 0$ while $P(H \cdot (X_t - X_{t-1}) > 0) > 0$. Note that this implies $Q(H \cdot (X_t - X_{t-1})) > 0$.

This cannot hold if H is bounded. We therefore consider $H^c := H \mathbb{1}_{\{H| \leqslant c\}}$ for $c > 0$. Since $\{H| \leqslant c\} \uparrow \Omega$ for $c \to \infty$, we can exploit σ-continuity of the probability measure Q. Hence $Q(H^{c^*}(X_t - X_{t-1}) > 0) \to Q(H(X_t - X_{t-1}) > 0) > 0$. Then there exists a c^* such that $Q(H^{c^*}(X_t - X_{t-1}) > 0) > 0$.

But,

$$E^Q[H^{c^*} \cdot (X_t - X_{t-1}) | \mathscr{F}_{t-1}] = H^{c^*} E^Q[X_t - X_{t-1} | \mathscr{F}_{t-1}] = 0,$$

which contradicts with $H^{c^*} \cdot (X_t - X_{t-1}) \geqslant 0$ and $Q(H^{c^*}(X_t - X_{t-1}) > 0) > 0$.

(i) $\Rightarrow$ (ii): Consider $Z \in (K - L^0_+) \cap L^0_+$. With a $\mathscr{F}_{t-1}$-measurable H and $U \in L^0_+$,

$$Z = H \cdot (X_t - X_{t-1}) - U \geqslant 0.$$

Hence $H \cdot (X_t - X_{t-1}) \geqslant U \geqslant 0$, such that $H \cdot (X_t - X_{t-1}) \in K \cap L^0_+$. By (i),t $H \cdot (X_t - X_{t-1}) = 0$, also $U = 0$ and so $Z = 0$.

The missing (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (iv) are immediate. $\square$

For the remaining step (ii) $\Rightarrow$ (iii) will be achieved through a number of steps:

(i) We first show that integrability can always achieved through a change of measure (Lemma 22). This allows us to consider the convex cone C as a subset of L^1, see (23).

(ii) Then we apply the Hahn-Banach theorem: consider $F \in L^1_+$ together with $B = \{F\}$ and C. The difficulty is to show that C is closed (which we postpone for a moment). This gives us a strictly separating continuous functional.

(iii) Using duality we obtain a density (Lemma 28 (i)). Since C is countably convex we can even select a positive density, hence an equivalent martingale measure (Lemma 28 (ii)).

(iv)

(v)

The following step shows that we can always achieve integrability by an equivalent measure change with a bounded density.

Lemma 22. *There exists $\widetilde{P} \sim P$ such that $\widetilde{E}[|X_t|] < \infty$ and $\widetilde{E}[|X_{t-1}|] < \infty$.*

Proof. Consider $c > 0$, let

$$Z := \frac{c}{1 + |X_t| + |X_{t-1}|} \leqslant c$$

and $d\widetilde{P} = Z dP$. Obviously, $\widetilde{E}[|X_t|] < \infty$ and $\widetilde{E}[|X_{t-1}|] < \infty$. $\square$

Since (ii) only depends on the nullsets of P, it holds if and only it if holds with respect to $\widetilde{P}$ when $\widetilde{P} \sim P$. The same holds for boundedness of the density and we therefore can assume without loss of generality that $E[|X_t|] < \infty$ und $E[|X_{t-1}|] < \infty$.

Define the convex cone

$$C = (K - L^0_+) \cap L^1. \tag{23}$$

Example 24 (C not closed)**.** It is remarkable that NA actually implies closedness of C: indeed, the following example (where an arbitrage exists) shows that C is not always closed: consider $\Omega = [0, 1]$, the Borel σ-field $\mathscr{F}_1 =$, trivial $\mathscr{F}_0 = \{\varnothing, \Omega\}$, and $\Delta X(\omega) = \omega$ (clearly, we have arbitrages here).

Note that C is a true subset of L^1, since for $F \geqslant 1$ (for all $\omega \in \Omega$), $F \notin C$. Define

$$F_n = (F^+ \wedge n)\mathbb{1}_{[1/n,1]} - F^- \quad \text{for } F \in L^1,$$

such that $F_n \xrightarrow{L^1} F$. Moreover, $F_n \in C$: note that

$$(F^+ \wedge n)\mathbb{1}_{[1/n,1]} \leqslant \begin{cases} n & \omega \in [1/n, 1], \\ 0 & \omega \in [0, 1/n). \end{cases}$$

Hence,

$$(F^+ \wedge n)\mathbb{1}_{[1/n,1]} \leqslant n \cdot n\Delta X = n^2 \Delta X,$$

such that $(F^+ \wedge n)\mathbb{1}_{[1/n,1]} = n^2\Delta X - U \in C$.

The Hahn-Banach theorem

Recall that a topological vector space is a vector space with a topology, such that addition and scalar multiplication are continuous. It is called *locally convex* if its topology is generated by convex sets. Note that any Banach space is locally convex, since we can use as base the ε balls for each element. However, the space of all random variable L^0 with the topology of convergence in probability is not convex if $(\Omega, \mathscr{F}, P)$ has no atoms.

Theorem 25 (Hahn-Banach). *Consider two non-empty sets B and C of the locally convex space E and assume that*

(i) $B \cap C = \varnothing$,

(ii) B, C *are convex,*

(iii) B *is compact and* C *is closed.*

Then there exists a continuous linear functional $\ell : E \to \mathbb{R}$, such that

$$\sup_{x \in C} \ell(x) < \inf_{x \in B} \ell(x).$$

By a duality argument the linear functional can be represented by a Z satisfying the following property (27). We first show that this is sufficient to obtain a martingale measure.

Lemma 26. *Let $c \geqslant 0$ and $Z \in L^\infty(P, \mathscr{F}_t)$, such that*

$$E[ZW] \leqslant c \qquad \text{for all } W \in C. \tag{27}$$

'Then

(i) $E[ZW] \leqslant 0$ *for all* $W \in C$,

(ii) $Z \geqslant 0$ *P-a.s.,*

(iii) If $P(Z > 0) > 0$, *then*

$$\frac{dQ}{dP} := Z$$

defines a martingale measure and $Q \ll P$.

Proof. (i) Since C is a cone, it follows for any $W \in C$ and $\alpha > 0$, that

$$E[ZW] = \alpha \cdot E[Z\alpha^{-1}W] \leqslant \alpha c,$$

and the equation holds already for $c = 0$.

(ii) Choose $W = -\mathbb{1}_{\{Z<0\}} \in C$. Then,

$$E[Z_-] = E[ZW] \leqslant 0,$$

such that $Z_- = 0$ and hence $Z \geq 0$ P-a.s.

(iii) Choose H in $L^\infty(P, \mathscr{F}_{t-1}, \mathbb{R}^d)$, $\alpha \in \mathbb{R}$ and $Y = (X_t - X_{t-1})$. Then, $Y \in C$ and, since H and Z are bounded,

$$E[ZHY] \leqslant c \quad \text{and} \quad E[\alpha ZHY] \leqslant c.$$

As above, $E[ZHY] \leqslant 0$. Außerdem ist für alle $\alpha \in \mathbb{R}$

$$\alpha E[ZHY] \leqslant 0,$$

also $E[ZHY] = 0 = E[H(X_t - X_{t-1})]$. Wir erhalten $E^Q[\mathbb{1}_F(X_t^i - X_{t-1}^i)] = 0$ für alle $F \in \mathscr{F}_{t-1}$, also ist X Q-Martingal. □

So, by Lemma 26, the existence of an equivalent martingale measure is equivalent to find an element of the following set:

$$\mathcal{Z} := \{Z \in L^\infty,\ 0 \leqslant Z \leqslant 1,\ P(Z > 0) > 0,\ E[ZW] \leqslant 0\ \forall W \in C\}.$$

Lemma 28. *Assume that C is closed in L^1 and that $C \cap L^1_+ = \{0\}$. Then,*

(i) for all $F \in L^1_+ \backslash \{0\}$ there exists an $Z \in \mathcal{Z}$, such that $E[FZ] > 0$, and

(ii) there exists $Z^* \in \mathcal{Z}$, such that $Z^* > 0$.

Proof. For the first part, consider $B = \{F\}$. Then $B \cap C = \varnothing$, $C \neq \varnothing$; both sets are convex, B is compact and C is closed by assumption.

Then, we can apply the theorem of Hahn-Banach which gives us a continuous linear functional ℓ, such that

$$\sup_{W \in C} \ell(W) < \ell(F). \tag{29}$$

The dual space of L^1 can be identified with L^∞ by the Riesz theorem, such the linear function ℓ can be represented by a $Z \in L^\infty$ such that

$$\ell(F') = E[Z \cdot F'], \qquad F' \in L^1.$$

Without loss of generality we may assume that $\|Z\|_\infty = 1$.

By Equation (29), we obtain that $\ell(W) = E[ZW] < \ell(F) = E[ZF]$ for all $W \in C$. Hence, Z satisfies (27) and we can apply Lemma 26. This yields that $Z \in \mathcal{Z}$. Since $0 \in C$ we obtain that $E[FZ] > 0$.

For the second part, we start by showing that $\mathcal{Z}$ is countably convex. To this choose $\alpha_k \in [0,1]$, $k \in \mathbb{N}$ with $\sum_{k=1}^\infty \alpha_k = 1$, and $(Z_k)_{k \in \mathbb{N}} \subset \mathcal{Z}$ and consider

$$Z := \sum_{k=1}^\infty \alpha_k Z_k.$$

For $W \in C$,

$$\sum_{k=1}^\infty |\alpha_k Z_k W| \leqslant |W| \sum_{k=1}^\infty |\alpha_k| = |W| \in L^1.$$

such that by dominated convergende

$$E[ZW] = \sum_{k=1}^\infty \alpha_k E[Z_k W] \leqslant 0$$

and hence $Z \in \mathcal{Z}$.

Now set

$$c := \sup\{P(Z > 0) :\ Z \in \mathcal{Z}\}.$$

and choose a sequence $(Z_n) \in \mathcal{Z}$, such that $P(Z_n > 0) \to c$. Then,

$$Z^* := \sum_{k=1}^{\infty} \frac{1}{2^k} Z_k \in \mathcal{Z}.$$

Now we show that $P(Z^* > 0) = 1$. Indeed, $\{Z^* > 0\} = \bigcup_{k=1}^{\infty} \{Z_k > 0\}$, such that

$$P(Z^* > 0) \geqslant \sup_{k \in \mathbb{N}} P(Z_k > 0) = c.$$

If we would have on the contrary that $P(Z^* = 0) > 0$, then $F := 1_{\{Z^*=0\}} \neq 0$ and $F \in L^1_+$. By Lemma 28 there exists $Z' \in \mathcal{Z}$, s.t.

$$0 < E[FZ'] = E[1_{\{Z^*=0\}} Z'].$$

Hence $P(\{Z' > 0\} \cap \{Z^* = 0\}) > 0$. This implies that the convex combination achieves

$$P\Big(\frac{1}{2}(Z' + Z^*) > 0\Big) > P(Z^* > 0),$$

a contradiction to the maximality of c and the claim follows. □

The following Lemma generalizes the theorem of *Bolzano-Weierstraß* to infinite dimensional spaces. Boundedness is not sufficient in infinite dimensional spaces, such that we require existence of an accumulation point instead.

Lemma 30. *Consider a sequence (H_n) of d-dimensional random variables and assume that*

$$\lambda := \liminf_n \parallel H_n \parallel < \infty.$$

Then there exists $H \in L^0(\mathbb{R}^d)$ and a strictly increasing sequence (σ_m) such that

$$H_{\sigma_m(\omega)}(\omega) \to H(\omega)$$

for P-almost all $\omega \in \Omega$.

The idea is to proceed pointwise, such that we can rely on the classical Bolzano-Weierstraß.

Proof. Define $\sigma_m = m$ on $\{\lambda = \infty\}$. On $\{\lambda < \infty\}$ let $\sigma_1^0 := 1$ and

$$\sigma_m^0(\omega) := \inf\Big\{n > \sigma_{m-1}^0(\omega) : \parallel H_n(\omega) \parallel - |\lambda(\omega)| \leqslant \frac{1}{m}\Big\} \qquad m = 2, 3, \ldots$$

Now we proceed inductively through the coordinates. We denote for the sequence (σ^{i-1}), $H^i := \liminf_{m\to\infty} H^i_{\sigma_m^{i-1}}$ and construct (σ^i) as follows: let $\sigma_1^i = 1$ and

$$\sigma_m^i(\omega) := \inf\Big\{\sigma_n^{i-1}(\omega) : \ \sigma_n^{i-1}(\omega) > \sigma_{m-1}^i(\omega) \quad \text{und} \quad |H^i_{\sigma_n^{i-1}(\omega)}(\omega) - H^i(\omega)| \leqslant \frac{1}{n}\Big\}.$$

Then, $\sigma_m := \sigma_m^d$ on $\{\lambda < \infty\}$ does the job. □

We are almost ready, but two portfolios can lead to the same payoff, which creates problems. Or, equivalently, it could happen that

$$H(X_t - X_{t-1}) = 0$$

holds even if $H \neq 0$. Using orthogonal projections we create a subset where this can not happen.

We consider the not locally convex space L^0 with the topology of convergence in probability, which is generated by the semi-metric $E[|X-Y| \wedge 1]$.

Lemma 31. *Define:*

$$N = \{H \in L^0(\Omega, \mathscr{F}_{t-1}, P; \mathbb{R}^d) : \; H(X_t - X_{t-1}) = 0 \quad P-f.s.\}$$
$$N^\perp = \{G \in L^0(\Omega, \mathscr{F}_{t-1}, P; \mathbb{R}^d) : \; G \cdot H = 0 \text{ für alle } H \in N\}$$

Then it holds that

(i) $N, N^\perp$ are closed in L^0. Moreover, for $g \in L^0(\Omega, \mathscr{F}_{t-1}, P; \mathbb{R})$ it holds that

$$gH \in N \text{ if } H \in N \quad \text{ and } gG \in N^\perp \text{ if } G \in N^\perp.$$

(ii) $N \cap N^\perp = \{0\}$.
(iii) Every $G \in L^0(\Omega, \mathscr{F}_{t-1}, P, \mathbb{R}^d)$ has the following unique decomposition

$$G = H + G^\perp, \quad H \in N, \; G^\perp \in N^\perp.$$

Proof. (i) Consider a sequence $H_n \xrightarrow{P} H$. Then we have an a.s. converging subsequence (H_{σ_m}). This implies

$$H_{\sigma_m}(\omega) \cdot (X_t(\omega) - X_{t-1})(\omega) \to H(\omega)(X_t(\omega) - X_{t-1}(\omega)) \quad \text{for } P\text{-almost all } \omega \in \Omega. \tag{32}$$

Similarly, if we now consider a subsequence $(H_n) \subseteq N$ with $H_n \to H$ a.s., then the left hand side of (32) is equal to 0 for all n and so is the limit. Hence, $H \in N$.

Similarly, for $(G_k) \subseteq N^\perp$ with $G_n \to^{a.s.} G$, we obtain $G \in N^\perp$.

The additional property is immediate.

(ii) Since for $G \in N \cap N^\perp$, it holds by definition that

$$0 = G \cdot G = |G|$$

which is equivalent to $G = 0$ a.s.

(iii) For $\xi \in \mathbb{R}^d$ we write $\xi = \xi^1 e_1 + \cdots + \xi^d e_d$ with respect to a basis $\{e_1, \ldots, e_d\}$.

First, assume that $e_i = n_i + e_i^\perp$ with $n_i \in N$ and $e_i^\perp \in N^\perp$, $i = 1, \ldots, d$. Then

$$\xi = \underbrace{\sum_{i=1}^d \xi n_i}_{\in N} + \underbrace{\sum_{i=1}^d \xi_i e_i^\perp}_{\in N^\perp}.$$

The decomposition is unique since $N \cap N^\perp = \{0\}$.

Now we show $e_i = n_i + e_i^\perp$. To this end consider the Hilbert space $L^2 = L^2(\Omega, \mathscr{F}_t, P; \mathbb{R}^d)$ with scalar product $\langle X, Y \rangle = E[XY]$. Both $N \cap L^2$ and $N^\perp \cap L^2$ are closed subspaces of L^2, since convergence in probability implies L^2-convergence and we already showed closeness of N and $N^\perp$. We define the orthogonal projections

$$\pi : L^2 \to N \cap L^2, \quad \pi^\perp : L^2 \to N^\perp \cap L^2$$

and set $n_i = \pi(e_i)$, $e_i^\perp = \pi^\perp(e_i)$.

Now consider $\xi := e_i - \pi(e_i)$. Since $\pi(e_i)$ is the orthogonal projection, it holds that

$$\langle \xi, n \rangle = 0 \tag{33}$$

for all $n \in N \cap L^2$. Note that $e_i \in L^2$ and so is $\pi(e_i)$, such that $\xi \in L^2$. We show that $\xi \in N^\perp$: assume, $\xi \notin N^\perp \cap L^2$. Then there exists $H \in N$ with $P(\xi \cdot H > 0) > 0$. Set

$$\widetilde{H} := H \mathbb{1}_{\{\xi \cdot H > 0\ ,\ |H| \leqslant c\}} \in N \cap L^2.$$

If c is large enough,

$$0 < E[\widetilde{H} \cdot \xi] = \langle \widetilde{H}, \xi \rangle,$$

a contradiction to (33).

□

The final step is done in the following lemma: it shows, that already $K \cap L^0_+ = \{0\}$ implies the required closeness.

Lemma 34. *If $K \cap L^0_+ = \{0\}$, then $K - L^0_+$ is closed in L^0.*

Proof. Consider a sequence (W_n) of elements of $K - L^0_+$ converging in L^0 (hence in probability) to W. By changing to a subsequence we can assume that the convergence is even almost surely. Then we have the representation

$$W_n = \widetilde{H}_n \cdot \Delta X - U_n \overset{L.31}{=} \widetilde{\widetilde{H}}_n \Delta X + H_n^\perp \Delta X - U_n = H_n^\perp \Delta X - U_n =: H_n \Delta X - U_n,$$

with $H_n \in N^\perp$, since $\widetilde{\widetilde{H}}_n \Delta X = 0$.

First, we assume that $\liminf |H_n| < \infty$ P-a.s. Then, Lemma 30 implies that we find a subsequence for which $H_{\sigma_n} \to H$ P-a.s. Moreover,

$$0 \leqslant U_{\sigma_n} = H_{\sigma_n} \Delta X - W_{\sigma_n} \to H \Delta X - W =: U \qquad P - \text{f.s.}$$

with some $U \geqslant 0$, such that $W \in K - L^0_+$ and closeness holds.

The proof is finished when we can show that $\liminf |H_n| < \infty$ P-a.s. To this end, consider the trading strategy $\xi_n = \frac{H_n}{|H_n|}$ and $A = \{\omega \in \Omega \ : \ \liminf |H_n| = \infty\}$. We apply 30 to $\xi_n = \frac{H_n}{|H_n|}$. This yields a subsequence (τ_n), such that $\xi_{\tau_n} \to \xi$ P-a.s. Now it holds that

$$0 \leqslant \mathbb{1}_A \frac{U_{\tau_n}}{|H_{\tau_n}|} = \mathbb{1}_A \left(\frac{H_{\tau_n}}{|H_{\tau_n}|} \cdot \Delta X - \frac{W_{\tau_n}}{|H_{\tau_n}|} \right) \to \mathbb{1}_A \xi \Delta X \quad P - \text{a.s.},$$

since $\frac{W_{\tau_n}}{|H_{\tau_n}|} \to 0$. This yield that $\mathbb{1}_A \xi \Delta X \in K \cap L^0_+$, such that by our assumption $\mathbb{1}_A \xi \Delta X = 0$.

Note that for $\eta \in N$,

$$\xi_{\tau_n} \cdot \eta = \sum_{k=1}^{\infty} \mathbb{1}_{\{\tau_n = k\}} \frac{1}{|H_k|} H_k \cdot \eta = 0,$$

since $H_k \in N^\perp$. Hence, $\xi_{\tau_n} \in N^\perp$ and so is $\mathbb{1}_A \xi_{\tau_n}$. Since $N^\perp$ is closed, $\mathbb{1}_A \xi \in N^\perp$. But we also showed that $\mathbb{1}_A \xi \in N$. This is, by (ii) of Lemma 30, only possible if $\mathbb{1}_A \xi = 0$. But $|\xi| = 1$, such that $P(A) = 0$ follows. □

The 2nd fundamental theorem

In this section we study the second fundamental theorem of asset pricing. A proof for this theorem in discrete time under trivial initial conditions can be found in Föllmer & Schied (2016). We follow Niemann & Schmidt (2024) and show a full proof of the conditional version using non-linear expectations.

We start with a super short introduction to non-linear expectations.

Dynamic non-linear expectations

As before, we consider a measurable space $(\Omega, \mathscr{F})$ with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in\{0,...,T\}}$. For this section, we assume that $\mathscr{F}_T = \mathscr{F}$ and that $\mathscr{F}_0$ is trivial.

Consider a set of probability measures $\mathscr{P}$ on $(\Omega, \mathscr{F})$. A P-null set $A \subseteq \Omega$ is a possibly not measurable set being a subset of a measurable set $A' \in \mathscr{F}$ with $P(A') = 0$. A set $A \subseteq \Omega$ is called a *$\mathscr{P}$-polar set*, if A is a P-null set for every $P \in \mathscr{P}$. We denote the collection of $\mathscr{P}$-polar sets by $\mathrm{Pol}(\mathscr{P})$. We say a property holds $\mathscr{P}$-quasi surely, in short $\mathscr{P}$-q.s., if it holds outside a $\mathscr{P}$-polar set. If $\mathscr{P} = \{P\}$, we write short $\mathrm{Pol}(P)$ instead of $\mathrm{Pol}(\{P\})$.

For two subsets of probability measures $\mathscr{P}$ and $\mathscr{Q}$, we call $\mathscr{Q}$ *absolutely continuous* with respect to $\mathscr{P}$, denoted by $\mathscr{Q} \ll \mathscr{P}$, if $\mathrm{Pol}(\mathscr{P}) \subseteq \mathrm{Pol}(\mathscr{Q})$. We write $\mathscr{Q} \sim \mathscr{P}$, if $\mathscr{Q} \ll \mathscr{P}$ and $\mathscr{P} \ll \mathscr{Q}$.

On $\mathscr{L}^0(\Omega, \mathscr{F}) = \{X : \Omega \to \mathbb{R} \colon X\ \mathscr{F}\text{-measurable}\}$ we introduce the equivalence relation $\sim_{\mathscr{P}}$ by $X \sim_{\mathscr{P}} Y$ if and only if $X = Y$ $\mathscr{P}$-q.s.. Then, we set

$$\begin{aligned}
L^0(\Omega, \mathscr{F}, \mathscr{P}) &:= \mathscr{L}^0(\Omega, \mathscr{F})/\mathscr{P} \\
L^p(\Omega, \mathscr{F}, \mathscr{P}) &:= \{X \in L^0(\Omega, \mathscr{F}, \mathscr{P}) : \sup_{P\in\mathscr{P}} E_P[|X|^p] < \infty\} \\
L^\infty(\Omega, \mathscr{F}, \mathscr{P}) &:= \{X \in L^0(\Omega, \mathscr{F}, \mathscr{P}) : \exists C > 0 : |X| \le C\ \mathscr{P}\text{-q.s.}\}
\end{aligned}$$

Then, Proposition 14 in Denis et al. (2011) shows that for each $p \in [1, \infty]$, $L^p(\Omega, \mathscr{F}, \mathscr{P})$ is a Banach space. The space $L^0(\Omega, \mathscr{F}, \mathscr{P})$ can be equipped with the metric d given by

$$d(X, Y) := \sup_{P\in\mathscr{P}} E_P[|X - Y| \wedge 1]$$

inducing uniform convergence in probability.

We consider a set $\mathscr{H} \subseteq L^0(\Omega, \mathscr{F}, \mathscr{P})$ containing all constants and set, for $t \in \{0, ..., T\}$,

$$\mathscr{H}_t := \mathscr{H} \cap L^0(\Omega, \mathscr{F}_t, \mathscr{P}).$$

The following definition introduces the notion of a conditional nonlinear expectation and the associated notion of a dynamic nonlinear expectation which is a set of conditional nonlinear expectations.

Definition 35. We call a mapping $\mathcal{E}_t : \mathscr{H} \to \mathscr{H}_t$ an *$\mathscr{F}_t$-conditional nonlinear expectation*, if

(i) $\mathcal{E}_t$ is *monotone*: for $X, Y \in \mathscr{H}$ the condition $X \le Y$ implies $\mathcal{E}_t(X) \le \mathcal{E}_t(Y)$,

(ii) $\mathcal{E}_t$ *preserves measurable functions*: for $X_t \in \mathscr{H}_t$ we have $\mathcal{E}_t(X_t) = X_t$.

We call $\mathcal{E} = (\mathcal{E}_t)_{t\in\{0,...,T\}}$ a *dynamic nonlinear expectation*, if for every $t \in \{0, ..., T\}$ the mapping $\mathcal{E}_t : \mathscr{H} \to \mathscr{H}_t$ is an $\mathscr{F}_t$-conditional nonlinear expectation.

We introduce further properties which will be of interest in the context of dynamic nonlinear expectations. First, we introduce some well-known properties regarding the set $\mathscr{H}$, all in an appropriate conditional formulation. Denote $\mathscr{H}_t^+ := \{X \in \mathscr{H}_t : X \ge 0\}$.

Definition 36. We call the set $\mathscr{H}$

(i) *symmetric*, if $-\mathscr{H} = \mathscr{H}$.

(ii) *additive*, if $\mathscr{H} + \mathscr{H} \subseteq \mathscr{H}$.

(iii) *$\mathscr{F}_t$-translation-invariant*, if $\mathscr{H} + \mathscr{H}_t \subseteq \mathscr{H}$.

(iv) *$\mathscr{F}_t$-convex*, if for $\lambda_t \in \mathscr{H}_t$ with $0 \leq \lambda_t \leq 1$ we have

$$\lambda_t \mathscr{H} + (1 - \lambda_t)\mathscr{H} \subseteq \mathscr{H}.$$

(v) *$\mathscr{F}_t$-positively homogeneous*, if $\mathscr{H}_t^+ \cdot \mathscr{H} \subseteq \mathscr{H}$.

(vi) *$\mathscr{F}_t$-local*, if $\mathbb{1}_A \mathscr{H} \subseteq \mathscr{H}$ for every $A \in \mathscr{F}_t$.

We simply call $\mathscr{H}$ *translation-invariant*, if it is $\mathscr{F}_t$-translation-invariant for every $t \in \{0, ..., T\}$ and do so in a similar fashion for all the other properties.

Next, we introduce well-known properties of nonlinear conditional expectations, all in an appropriate conditional formulation which are frequently used for example in the context of risk measures.

Definition 37. An $\mathscr{F}_t$-conditional expectation $\mathcal{E}_t$ is called

(i) *subadditive*, if $\mathscr{H}$ is additive

$$\mathcal{E}_t(X + Y) \leq \mathcal{E}_t(X) + \mathcal{E}_t(Y), \qquad X,\ Y \in \mathscr{H}.$$

(ii) *$\mathscr{F}_t$-translation-invariant*, if $\mathscr{H}$ is $\mathscr{F}_t$-translation-invariant and

$$\mathcal{E}_t(X + X_t) = \mathcal{E}_t(X) + X_t, \qquad X \in \mathscr{H},\ \ X_t \in \mathscr{H}_t$$

(iii) *$\mathscr{F}_t$-convex*, if $\mathscr{H}$ is $\mathscr{F}_t$-convex and

$$\mathcal{E}_t\left(\lambda_t X + (1 - \lambda_t)Y\right) \leq \lambda_t \mathcal{E}_t(X) + (1 - \lambda_t)\mathcal{E}_t(Y), \quad 0 \leq \lambda_t \leq 1,\ \lambda_t \in \mathscr{H}_t,\ X, Y \in \mathscr{H}.$$

(iv) *$\mathscr{F}_t$-positively homogeneous*, if $\mathscr{H}$ is $\mathscr{F}_t$-positively homogeneous and

$$\mathcal{E}_t(X_t X) = X_t \mathcal{E}_t(X), \qquad X \in \mathscr{H},\ X_t \in \mathscr{H}_t^+.$$

(v) *$\mathscr{F}_t$-sublinear*, if it is subadditive and $\mathscr{F}_t$-positively homogeneous.

(vi) *$\mathscr{F}_t$-local*, if $\mathscr{H}$ is $\mathscr{F}_t$ local and

$$\mathcal{E}_t(\mathbb{1}_A X) = \mathbb{1}_A \mathcal{E}_t(X), \qquad X \in \mathscr{H},\ A \in \mathscr{F}_t.$$

Moreover, we call a dynamic expectation $\mathcal{E} = (\mathcal{E}_t)_{t \in \{0,...,T\}}$ translation-invariant, (subadditive, convex or positively homogeneous) if for every $t \in \{0, ..., T\}$ the $\mathscr{F}_t$-conditional expectation $\mathcal{E}_t$ has the corresponding property.

Sensitivity and time consistency

In contrast to a classical expectation, a nonlinear expectation might contain only little information on underlying random variables. Sensitivity is a property which allows at least to separate zero from positive random variables. It should be noted that such sensitivity on a suitable set of random variables is implied by no-arbitrage.

Definition 38. We call an $\mathscr{F}_t$-conditional nonlinear expectation $\mathcal{E}_t$ *sensitive*, if for every $X \in \mathscr{H}$ with $X \geq 0$ and $\mathcal{E}_t(X) = 0$ we have $X = 0$.

Similarly, we call the dynamic nonlinear expectation $\mathcal{E}$ *sensitive*, if all $\mathcal{E}_t$, $t = 0, \dots, T$ are sensitive.

Time consistency is an important property in the context of dynamic risk measures, which has been intensively studied. It transports the tower-property to the non-linear setting.

Definition 39. We call a dynamic expectation $\mathcal{E}$ *time-consistent*, if

$$\mathcal{E}_s = \mathcal{E}_s \circ \mathcal{E}_t, \qquad 0 \le s \le t \le T.$$

Since $\mathscr{F} = \mathscr{F}_T$, $\mathcal{E}_T$ is the identity and hence, every expectation is time-consistent between $T-1$ and T, i.e.,

$$\mathcal{E}_{T-1} \circ \mathcal{E}_T = \mathcal{E}_{T-1}.$$

Remark 40 (Extension of time-consistency to stopping times)**.** For simplicity, we restrict our definition of time consistency to deterministic times $s, t \in \{0, \dots, T\}$. This can easily be generalized when $\mathcal{E}$ is translation-invariant and local: indeed, let τ be a stopping time with values in $\{0, \dots, T\}$. Given $(\mathcal{E}_t)_t$, we define

$$\mathcal{E}_\tau(H) := \sum_s \mathbb{1}_{\{\tau = s\}} \mathcal{E}_s(H).$$

If $(\mathcal{E}_t)_t$ is time-consistent, then for any two such stopping times σ, τ with $\sigma \le \tau$, the equality

$$\mathcal{E}_\sigma \circ \mathcal{E}_\tau = \mathcal{E}_\sigma$$

holds whenever $\mathcal{E}$ is translation-invariant and local.

The remarkable connection between sensitivity and time consistency can already be seen from the simple observation that a time-consistent dynamic expectation is already sensitive, if $\mathcal{E}_0$ is sensitive.

Remark 41. Let $\mathscr{P}$ and $\mathscr{Q}$ be two sets of probability measures on $(\Omega, \mathscr{F})$, and let $\mathcal{E}_t : L^\infty(\Omega, \mathscr{F}, \mathscr{Q}) \to L^\infty(\Omega, \mathscr{F}_t, \mathscr{Q})$ be a conditional nonlinear expectation. Then, $\mathcal{E}_t$ is well-defined on $L^\infty(\Omega, \mathscr{F}, \mathscr{P})$ if and only if $\mathscr{Q} \ll \mathscr{P}$. However, for $H \in L^\infty(\Omega, \mathscr{F}, \mathscr{P})$ the evaluation $\mathcal{E}_t(H)$ is a priori only an element of $L^\infty(\Omega, \mathscr{F}_t, \mathscr{Q})$. For it to be well-defined in $L^\infty(\Omega, \mathscr{F}, \mathscr{P})$ we require $\mathscr{Q} \sim \mathscr{P}$ on $\mathscr{F}_t$. Hence, if $\mathscr{Q} \ll \mathscr{P}$ and $\mathscr{Q} \sim \mathscr{P}$ on $\mathscr{F}_t$, the conditional nonlinear expectation $\mathcal{E}_t$ induces a nonlinear expectation $\bar{\mathcal{E}}_t : L^\infty(\Omega, \mathscr{F}, \mathscr{P}) \to L^\infty(\Omega, \mathscr{F}_t, \mathscr{P})$. In case $\mathcal{E}_t$ is sensitive, sensitivity of $\bar{\mathcal{E}}_t$ is equivalent to $\mathscr{P} \sim \mathscr{Q}$.

Lemma 42 below generalizes the well-known result that the acceptance sets of time-consistent expectations are decreasing: if $\mathcal{E}$ is time-consistent, then

$$\{\mathcal{E}_s \le 0\} \supseteq \{\mathcal{E}_t \le 0\}$$

for $s \le t$.

Lemma 42. *Let $\mathcal{E}$ be a time-consistent, local dynamic nonlinear expectation, fix $t \in \{0, \dots, T\}$ and consider $H \in \mathscr{H}$. If $\mathcal{E}_t(H) \le 0$, then*

$$\mathcal{E}_s(\mathbb{1}_A H) \le 0 \qquad \text{for all } A \in \mathscr{F}_t \text{ and all } s \le t.$$

If $\mathcal{E}_0$ is sensitive, then $\mathcal{E}_s(\mathbb{1}_A H) \le 0$ for all $A \in \mathscr{F}_t$ and some $s \le t$ implies that $\mathcal{E}_t(H) \le 0$.

Proof. Since $\mathcal{E}$ is local, $\mathcal{E}_t(H \mathbb{1}_A) = \mathbb{1}_A \mathcal{E}_t(H) \le 0$. Together with monotonicity we obtain

$$\mathcal{E}_s(\mathbb{1}_A H) = \mathcal{E}_s(\mathbb{1}_A \mathcal{E}_t(H)) \le 0.$$

Now, suppose $\mathcal{E}_0$ is sensitive. If $s = t$, the result is clear with $A = \Omega$. Let $s < t$ and note that $\mathcal{E}_s$ is sensitive. To show that $\mathcal{E}_t(H) \leq 0$, it thus suffices to show

$$\mathcal{E}_s(\mathbb{1}_A \mathcal{E}_t(H)) = 0$$

for $A := \{\mathcal{E}_t(H) \geq 0\} \in \mathscr{F}_t$. However, as above,

$$\mathcal{E}_s(\mathbb{1}_A \mathcal{E}_t(H)) = \mathcal{E}_s(\mathbb{1}_A H)$$

and the latter vanishes by assumption. □

Let $\mathcal{E}_0^*$ be a $\mathscr{F}_0$-conditional expectation. A *dynamic extension* of $\mathcal{E}_0^*$ is a dynamic expectation $\mathcal{E}$ such that $\mathcal{E}_0 = \mathcal{E}_0^*$.

Note that for any collection $\mathscr{P}$ of probability measures, the associated nonlinear expectation $\sup_{P \in \mathscr{P}} E_P[\cdot]$ is sensitive; see Remark 41. We call $\mathscr{P}$ *dominated* if there exists a probability measure P on $(\Omega, \mathscr{F})$ with $\mathscr{P} \ll \{P\}$, i.e., every P-null set is $\mathscr{P}$-polar. In this case, the Halmos-Savage Lemma guarantees the existence of a countable collection $\{P_n : n \in \boldsymbol{N}\} \subseteq \mathscr{P}$ with $\{P_n : n \in \boldsymbol{N}\} \sim \mathscr{P}$. In particular, there exists a measure P^* (not necessarily contained in $\mathscr{P}$) such that $\mathscr{P} \sim P^*$. Consequently, for any set of random variables $M \subseteq L^0(\Omega, \mathscr{F}, \mathscr{P}) = L^0(\Omega, \mathscr{F}, P^*)$, there exists a random variable called the $\mathscr{P}$-essential infimum and denoted by $\mathscr{P} - \operatorname{ess\,inf} M$ such that

(i) $\mathscr{P} - \operatorname{ess\,inf} M \leq Y$ $\mathscr{P}$-q.s. for every $Y \in M$,

(ii) $\mathscr{P} - \operatorname{ess\,inf} M \geq Z$ $\mathscr{P}$-q.s. for every random variable Z satisfying $Z \leq Y$ $\mathscr{P}$-q.s. for every $Y \in M$.

If $\mathscr{P}$ is not dominated, the $\mathscr{P}$-essential infimum might not exist, and it has in general no countable representation. In light of the financial applications we have in mind, we will assume in the next lemma that $\mathscr{P}$ is dominated.

Lemma 43. *Assume that $\mathscr{P}$ is dominated. Then, every sensitive $\mathscr{F}_0$-conditional expectation $\mathcal{E}_0$ on a symmetric set $\mathscr{H}$ has at most one translation-invariant, local, time-consistent dynamic extension $\mathcal{E}$. If it exists, it is given by*

$$\mathcal{E}_t(H) = \mathscr{P} - \operatorname{ess\,inf}\{H_t \in \mathscr{H}_t : H - H_t \in A_t\},$$

where

$$A_t := \{H \in \mathscr{H} : \mathcal{E}_0(\mathbb{1}_A H) \leq 0,\ \forall A \in \mathscr{F}_t\}.$$

Proof. Lemma 42 characterizes for $t \geq 1$ the acceptance set $A_t := \{H \in \mathscr{H} : \mathcal{E}_t \leq 0\}$ solely in terms of $\mathcal{E}_0$ and $\boldsymbol{F}$. Indeed, it yields that

$$A_t = \{H \in \mathscr{H} : \mathcal{E}_0(\mathbb{1}_A H) \leq 0\ \forall A \in \mathscr{F}_t\}.$$

This allows to recover every translation-invariant nonlinear expectation on a symmetric set from its acceptance set through the representation

$$\begin{aligned}\mathcal{E}_t(H) &= \mathscr{P} - \operatorname{ess\,inf}\{H_t \in \mathscr{H}_t : H_t \geq \mathcal{E}_t(H)\}\\ &= \mathscr{P} - \operatorname{ess\,inf}\{H_t \in \mathscr{H}_t : H - H_t \in A_t\}.\end{aligned}$$

Summarizing, we have obtained an explicit expression of the extension. □

Next, we verify that translation-invariance implies locality if $\mathscr{H} \subseteq L^\infty(\Omega, \mathscr{F}, \mathscr{P})$. This implies that every conditional risk measure on $L^\infty(\Omega, \mathscr{F}, \mathscr{P})$ is local. In particular, for every probability measure P, every dynamic risk-measure on $L^\infty(\Omega, \mathscr{F}, P)$ has at most one time-consistent extension. Moreover, one can show that not every coherent risk measure has a time-consistent extension.

Proposition 44. *Every translation-invariant expectation on a local set $\mathscr{H} \subseteq L^\infty(\Omega, \mathscr{F}, \mathscr{P})$ is local.*

Proof. Let $\mathcal{E}_t$ be translation-invariant and $A \in \mathscr{F}_t$. Further, let $H \in \mathscr{H}$. The inequality

$$\mathbb{1}_A H - \mathbb{1}_{A^c} \|H\|_\infty \leq H \leq \mathbb{1}_A H + \mathbb{1}_{A^c} \|H\|_\infty,$$

yields

$$\mathcal{E}_t(H) \geq \mathcal{E}_t(\mathbb{1}_A H - \mathbb{1}_{A^c} \|H\|_\infty),$$

and additionally

$$\mathcal{E}_t(H) \leq \mathcal{E}_t(\mathbb{1}_A H + \mathbb{1}_{A^c} \|H\|_\infty).$$

Multiplying both inequalities with $\mathbb{1}_A$ gives, exploiting translation invariance,

$$\mathbb{1}_A \mathcal{E}_t(H) = \mathbb{1}_A \mathcal{E}_t(\mathbb{1}_A H),$$

and thus

$$\begin{aligned}
\mathcal{E}_t(\mathbb{1}_A H) &= \mathbb{1}_A \mathcal{E}_t(\mathbb{1}_A H) + \mathbb{1}_{A^c} \mathcal{E}_t(\mathbb{1}_A H) \\
&= \mathbb{1}_A \mathcal{E}_t(H) + \mathbb{1}_A \mathbb{1}_{A^c} \mathcal{E}_t(\mathbb{1}_A H) \\
&= \mathbb{1}_A \mathcal{E}_t(H).
\end{aligned}$$

□

Super- and Sub-hedging

Now we turn back to a financial market. Recall that we worked on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, P)$ with a fixed probability measure P. Our aim is to study the upper bound of the set of no-arbitrage prices in more detail. It is given by

$$\bar{\mathcal{E}}_t(C_T) := \operatorname{esssup}\{E_Q[C_T \mid \mathscr{F}_t] : Q \in \mathscr{M}_e\}, \tag{45}$$

for $H \in L^\infty(P)$. We are interested in its relation to the smallest super-hedging price given by

$$\mathcal{E}_t(C_T) := \operatorname{ess\,inf}\{C_t \in L^0_t : \exists\, H \in \text{Pred} : C_t + G_t(H) \geq C_T\}, \tag{46}$$

where the gains process for the predictable (and hence self-financing) strategy H is given by $G_t(H) := (H \cdot X)_T - (H \cdot X)_t$. We denote $\mathcal{E}(C)$ for the process $(\mathcal{E}_t(C))_{t \in \{0,t\dots,T\}}$ and use a similar notation for the other dynamic non-linear expectations.

Lemma 47. *Assume that NA holds. Then, the super-hedging price $\mathcal{E}_t$ defined in* (46) *is a sensitive and $\mathscr{F}_t$-sub-linear, $\mathscr{F}_t$-conditional nonlinear expectation on L^∞ for all $0 \leq t \leq T$.*

Proof. First, we show that $\mathcal{E}_t(C_T)$ is bounded for $C_T \in L^\infty(P)$. The inequality $\mathcal{E}_t(C_T) \leq \|C_T\|$ follows by definition. If $C_t \in L^0_t$ is a superhedging price, choose $H \in$ Pred with $C_t + G_t(H) \geq C_T$.

Consider the set $A := \{C_t < -\|C_T\|\} \in \mathscr{F}_t$. Then there exists $H \in$ Pred with $G_t(C_T) \geq \mathbb{1}_A(C_T - C_t) \geq 0$ and therefore $P(A) = 0$ by absence of arbitrage. We conclude that $-\|C_T\| \leq \mathcal{E}_t(C_T) \leq \|C_T\|$.

Second, one easily checks the properties of a sublinear expectation. The sensitivity of $\mathcal{E}_t$ follows from the no-arbitrage assumption. □

Up to now we treated only bounded random variables, which excludes for example European calls. Typically one would consider the space $L^0_+(\mathscr{F}_T)$, which is of course not symmetric. The extension to $L^0(\mathscr{F}_T)$ is done by establishing continuity from below of the superheding price. The main argument resides of course on monotone convergence.

The superhedging duality

At time $t < T$, an $\mathscr{F}_t$ - measurable random variable π_t is a superhedging price for the European claim C_T due at time T, if there is a self-financing trading strategy which provides always a terminal wealth greater than C_T, i.e.there exists a predictable process H, such that

$$\pi_t + G_t(H) \geq C_T.$$

Remark 48. For every $C_T \in L^0(\mathscr{F}_T)$, and every predictable H one has the orthogonality

$$\mathcal{E}_t(C_T + G_t(H)) = \mathcal{E}_t(C_T). \tag{49}$$

For the following lemma we recall that on a upwards directed set, i.e. for $X, Y \in M$ there exists $Z \in M$ such that $Z \geq X \vee Y$, the essential supremum can be approximated by a sequence, see for example (Föllmer & Schied 2016, Theorem A.37). The lemma shows that the essential infimum of all superhedging prices is itself a superhedging price.

Lemma 50. *Assume that NA holds. For every $C_T \in L^0(\mathscr{F}_T)$, $\mathcal{E}_t(C_T)$ is a superhedging price for C_T.*

Proof. The set $M := \{C_t \in L^0_t : \exists H \in \text{Pred} : C_t + G_t(H) \geq C_T\}$ of superhedging prices is directed downwards. Hence, by (Föllmer & Schied 2016, Theorem A.37) there exists a sequence $(C^n_t)_n \subseteq M$ with $C^n_t \downarrow \mathcal{E}_t(C)$ a.s. By construction, we may write for each $n \in \mathbb{N}$,

$$C_T = C^n_t + G_t(H^n) - U^n$$

for some $H^n \in \text{Pred}$ and $U^n \in L^0_+(\mathscr{F}_T)$.

As the cone $\{G_t(H) - U : H \in \text{Pred}, U \in L^0_+(\mathscr{F}_T)\}$ is closed by Lemma 34, the claim follows. □

The following result shows that the superhedging prices are actually time-consistent.

Theorem 51. *Assume that NA holds. Then, the dynamic non-linear expectation $\mathcal{E}$ is time-consistent on L^∞.*

Proof. Applying Lemma 50 to the European contingent claims $\mathcal{E}_{t+1}(C_T)$ and C_T allows to choose strategies $H, H' \in \text{Pred}$ such that

$$\mathcal{E}_t(\mathcal{E}_{t+1}(C_T)) + G_t(H) \geq \mathcal{E}_{t+1}(C_T)$$

and

$$\mathcal{E}_{t+1}(C_T) + G_{t+1}(H') \geq C_T.$$

Combining both inequalities yields

$$\mathcal{E}_t(\mathcal{E}_{t+1}(C_T)) + G_t(H) + G_{t+1}(H') \geq C_T.$$

Hence, the claim C_T can be super-replicated at time t at price $\mathcal{E}_t(\mathcal{E}_{t+1}(C_T))$. As $\mathcal{E}_t(C_T)$ is by definition the smallest super-hedging price for claim C_T at time t, we obtain

$$\mathcal{E}_t(\mathcal{E}_{t+1}(C_T)) \geq \mathcal{E}_t(C_T).$$

Next, we obtain from Lemma 50 the existence of $H'' \in \text{Pred}$, such that

$$\mathcal{E}_t(C_T) + G_t(H'') \geq C_T.$$

Applying $\mathcal{E}_{t+1}$ to this inequality gives

$$\mathcal{E}_t(C_T) + \mathcal{E}_{t+1}(G_t(H'')) \geq \mathcal{E}_{t+1}(C_T).$$

By Equation (49),

$$\mathcal{E}_{t+1}(G_t(H'')) = H''_{t+1}\Delta X_{t+1} + \mathcal{E}_{t+1}(G_{t+1}(H'')) = H''_{t+1}\Delta X_{t+1}$$

and hence

$$\mathcal{E}_t(C_T) + H''_{t+1}\Delta X_{t+1} \geq \mathcal{E}_{t+1}(C_T).$$

Again, by Equation (49),

$$\mathcal{E}_t(\mathcal{E}_{t+1}(C_T)) \leq \mathcal{E}_t(C_T)$$

and the result is proven. □

Remark 52 (Time-consistency of $\bar{\mathcal{E}}$)**.** Consistency of $\bar{\mathcal{E}}$ is related to the stability of $\mathscr{M}_e$. With a little bit of work we obtain from (Föllmer & Schied 2016, Theorem 11.22) that the expectation $\bar{\mathcal{E}}$ is time-consistent.

The following result is the famous super-hedging duality.

Corollary 53 (Superhedging duality on L^∞)**.** *Assume (NA) holds. Then, for every $0 \leq t \leq T$ and every $C_T \in L^\infty(P)$, the superhedging-duality*

$$\mathcal{E}_t(C_T) = \bar{\mathcal{E}}_t(C_T) \tag{54}$$

holds.

Proof. By Theorem 51 and Remark 52, both $\mathcal{E}$ and $\bar{\mathcal{E}}$ are time-consistent. Moreover, they are translation invariant and hence local by Proposition 44. We leave the claim that

$$\mathcal{E}_0 = \bar{\mathcal{E}}_0$$

to the reader. Then, Lemma 43 implies the claim. □

By some monotone convergence arguments this can be extended to the space of claims (i.e. non-negative random variables).

Proposition 55 (Superhedging duality for claims)**.** *Assume that (NA) holds. The superhedging-duality* (54)*, and consistency of $\mathcal{E}$ extends to $L^0_+(\mathscr{F}_T)$.*

For the proof we refer to Proposition 2.16 in Niemann & Schmidt (2024).

Next, we prove a version of the optional decomposition directly by relying on the superheding-duality. For the stochastic integral until t we use the following notation

$$(H \cdot X)_t = \sum_{s=1}^{t} H_s \, \Delta X_s.$$

Theorem 56 (Optional decomposition)**.** *Assume that (NA) holds and let V be a non-negative $\mathscr{M}_e$-supermartingale. Then there exists an adapted increasing process C with $C_0 = 0$, and a predictable process H such that*

$$V_t = V_0 + (H \cdot X)_t - C_t \,.$$

Proof. By assumption,

$$E_Q[V_t \mid \mathscr{F}_{t-1}] \leq V_{t-1}$$

for every $0 \leq t \leq T$ and $Q \in \mathscr{M}_e$. This is equivalent to $\bar{\mathcal{E}}_{t-1}(V_t) \leq V_{t-1}$ and hence $\bar{\mathcal{E}}_{t-1}(\Delta V_t) \leq 0$. By Proposition 55,

$$\mathcal{E}_{t-1}(\Delta V_t) \leq 0.$$

Hence, for $t \in \{1, \dots, T\}$ there exists a strategy $H = H^{(t)} \in \text{Pred}$ such that

$$\Delta V_t \leq G_{t-1}(H) = \sum_{s=t}^{T} H_s \Delta X_s.$$

an application of $\mathcal{E}_t$ on both sides yields, by Equation (49),

$$\mathcal{E}_t(\Delta V_t) = \Delta V_t \leq \mathcal{E}_t(H_t \Delta X_t + G_t(H)) = H_t \Delta X_t.$$

Summing over $t \in \{0, \dots, T\}$, we obtain a predictable H' such that $(H' \cdot X) - V$ is increasing. $\square$

Structure of arbitrage-free prices

The main goal in computing arbitrage-free prices relying on the fundamental theorem of asset pricing is to obtain a price process for a new security which can be added to the market without violating absence of arbitrage.

In this spirit, an $\mathscr{F}_t$-measurable random variable π_t is called *arbitrage-free price* (at time t) of a European contingent claim C_T if there exists an adapted process X^{d+1} such that $X^{d+1}_t = \pi_t$, $X^{d+1}_T = H$ and the market (X, X^{d+1}) extended with X^{d+1} is free of arbitrage. Note that everything is formulated in discounted terms here. Denote by $\Pi_t(C_T)$ the collection of arbitrage free prices at time t.

Denote the upper and the lower bound of the no-arbitrage set at time t by

$$\pi^{\sup}_t(C_T) := \operatorname{esssup} \Pi_t(C_T), \quad \text{and} \quad \pi^{\inf}_t(C_T) := \operatorname{ess\,inf} \Pi_t(C_T).$$

To achieve countable convexity of the set of equivalent martingale measures we exploit nonnegativity of the price process and triviality of the initial σ-algebra $\mathscr{F}_0$ in the following lemma.

Lemma 57. *$\mathscr{M}_e$ is countably convex.*

Proof. Let $(Q^n) \subseteq \mathscr{M}_e$ and $(\lambda^n)_n \subseteq \mathbb{R}_+$ with $\sum_n \lambda^n = 1$. Set $Q^* := \sum_n \lambda^n Q^n$. Obviously, $Q^* \sim P$. For every $t \in \{1, \dots, T\}$ we have by monotone convergence

$$E_{Q^*}[X_t] = \sum_n \lambda^n E_{Q^n}[X_t] = \sum_n \lambda^n X_0 = X_0 < \infty$$

and hence $X_T \in L^1(Q^*)$. Similarly, for $A \in \mathscr{F}_{t-1}$,

$$E_{Q^*}[X_t \mathbb{1}_A] = \sum_n \lambda^n E_{Q^n}[X_t \mathbb{1}_A] = \sum_n \lambda^n E_{Q^n}[X_{t-1} \mathbb{1}_A] = E_{Q^*}[X_{t-1} \mathbb{1}_A]$$

and therefore $E^*[X_t \mid \mathscr{F}_{t-1}] = X_{t-1}$. $\square$

It is important to acknowledge that, in the notation of Lemma 57, we typically do not have

$$E_{Q^*}[H \mid \mathscr{F}_t] = \sum_n \lambda^n E_{Q^n}[H \mid \mathscr{F}_t]$$

for bounded H at $t > 0$, while this holds, as just shown, for X^i_T, $i = 1, \dots, d$.

Proposition 58. *For every $t \in \{0, \dots, T\}$, and for every $C_T \in L^0_+(\mathscr{F}_T)$ the set*

$$\{E_Q[C_T \mid \mathscr{F}_t] : Q \in \mathscr{M}_e\} \tag{59}$$

is $\mathscr{F}_t$-countably convex.

Proof. Let $(Q^n) \subseteq \mathscr{M}_e$. By pasting we may assume that all Q^n agree on $\mathscr{F}_t$.

Set $Q^* := \sum_n 2^{-n} Q^n$. By Lemma 57, $Q^* \in \mathscr{M}_e$. Denote by $Z^n := dQ^n/dQ^*$ the associated densities. As $Q^* = Q^n$ on $\mathscr{F}_t$ for each $n \in \mathbf{N}$, we have

$$Z^n_t = E_{Q^*}[Z^n \mid \mathscr{F}_t] = 1.$$

Since $C_T \geq 0$, monotone convergence implies for a sequence $(\lambda^n_t) \in L^0_+(\mathscr{F}_t)$ with $\sum_n \lambda_t = 1$, that

$$\sum_n \lambda^n_t E_{Q^n}[C_T \mid \mathscr{F}_t] = E_{Q^*}\Big[C_T \sum_n \lambda^n_t Z^n \mid \mathscr{F}_t\Big].$$

Set $Z := \sum_n \lambda^n_t Z^n > 0$. Note that

$$E_{Q^*}\Big[\sum_n \lambda^n_t Z^n \mid \mathscr{F}_t\Big] = \sum_n \lambda^n_t = 1$$

and we may therefore define the measure Q by

$$dQ/dQ^* := Z.$$

Then,

$$E_{Q^*}\Big[C_T \sum_n \lambda^n_t Z^n \mid \mathscr{F}_t\Big] = E_Q[C_T \mid \mathscr{F}_t].$$

It remains to verify that Q is indeed a martingale measure (after t). As the price process is nonnegative, its conditional expectation is well-defined, and we obtain by monotone convergence for $s \geq t$

$$\begin{aligned}
E_Q[X_{s+1} \mid \mathscr{F}_s] &= E_{Q^*}\Big[\frac{Z}{Z_s} X_{s+1} \mid \mathscr{F}_s\Big] \\
&= \frac{1}{Z_s} \sum_n \lambda^n_t E_{Q^*}[Z^n X_{s+1} \mid \mathscr{F}_s] \\
&= \frac{1}{Z_s} \sum_n \lambda^n_t Z^n_s E_{Q^n}[X_{s+1} \mid \mathscr{F}_s] = X_s
\end{aligned}$$

such that $Q \in \mathscr{M}_e$. □

Note that, due to the integrability conditions, $\Pi_t(C_T)$ is not necessarily $\mathscr{F}_t$-countably convex. Even in the unconditional case this fails. It is an easy consequence that integrability is the only difference between the set of risk-neutral expectations in (59) and $\Pi_t(C_T)$.

Lemma 60. *Consider $Q \in \mathscr{M}_e$. If $E_Q[C_T \mid \mathscr{F}_t]$ is finite-valued, then it is an arbitrage-free price.*

Proof. If $\xi := E_Q[C_T \mid \mathscr{F}_t]$ is finite, it is an element of $L^0_+(\mathscr{F}_T)$. We recall that we always may achieve integrability for ξ under an equivalent martingale measure: indeed, we can always find a P' such that $\xi \in L^1(P)$. Then we can choose a martingale measure with a bounded density.

Hence, there exists $\widetilde{Q} \in \mathscr{M}_e$ such that $E_Q[C_T \mid \mathscr{F}_t]$ is integrable with respect to $\widetilde{Q}$. We now paste Q and $\widetilde{Q}$, which is again a martingale measure. By construction, we even have $\widetilde{Q} \odot_t Q \in \mathscr{M}_e^{C_T}$. Moreover, it follows that

$$E_{\widetilde{Q}\odot_t Q}[C_T \mid \mathscr{F}_t] = E_Q[C_T \mid \mathscr{F}_t].$$

Since the associated price process is a martingale, this an arbitrage-free price by the fundamental theorem 15. □

Corollary 61. *Consider a claim $C_T \in L^0_+(\mathscr{F}_T)$, let $(Q^n) \subseteq \mathscr{M}_e^H$ and $(\lambda^n_t) \subseteq L^0_+(\mathscr{F}_t)$ with $\sum_n \lambda^n_t = 1$. If $\sum_n \lambda^n_t E_{Q^n}[C_T \mid \mathscr{F}_t]$ is finite-valued, it is contained in $\Pi_t(C_T)$.*

Proof. Due to Proposition 58, there exists $Q \in \mathscr{M}_e$ with

$$\sum_n \lambda^n_t E_{Q^n}[C_T \mid \mathscr{F}_t] = E_Q[C_T \mid \mathscr{F}_t]\,.$$

Now the claim follows by Lemma 60. □

Now we collect some properties of the non-linear expectation Π_t.

Corollary 62. *Consider $t \in \{0, \dots, T\}$. Then*

(i) $\Pi_t(H)$ *is $\mathscr{F}_t$-convex for every claim $H \in L^0_+(\mathscr{F}_T)$,*

(ii) $\Pi_t(H)$ *is directed upwards for every claim $H \in L^0_+(\mathscr{F}_T)$,*

(iii) $\Pi_t(H)$ *is $\mathscr{F}_t$-countably convex for every bounded claim $H \in L^\infty(P)$, and,*

(iv) for $H \in L^0_+(\mathscr{F}_T)$, any partition $(A^n) \subseteq \mathscr{F}_t$, and any sequence $(Q^n) \subseteq \mathscr{M}_e^H(P)$,

$$\sum_n \mathbb{1}_{A^n} E_{Q^n}[H \mid \mathscr{F}_t] \in \Pi_t(H)\,.$$

The next step is to show that Π_t is also local.

Lemma 63. *For $t \in \{0, \dots, T\}$, $A \in \mathscr{F}_t$ and $H \in L^0_+(\mathscr{F}_T)$, it holds that*

$$\Pi_t(\mathbb{1}_A H) = \mathbb{1}_A \Pi_t(H).$$

Proof. Let $Q \in \mathscr{M}_e$ such that $H\mathbb{1}_A$ is integrable with respect to Q. By construction $E_Q[H \mid \mathscr{F}_t]\mathbb{1}_A + E_{\widetilde{Q}}[H \mid \mathscr{F}_t]\mathbb{1}_{A^c}$ is finite, and by Corollary 62 and Lemma 60 there exists $Q^* \in \mathscr{M}_e^H$ with

$$E_{Q^*}[H \mid \mathscr{F}_t] = E_Q[H \mid \mathscr{F}_t]\mathbb{1}_A + E_{\widetilde{Q}}[H \mid \mathscr{F}_t]\mathbb{1}_{A^c}$$

and therefore

$$E_{Q^*}[H \mid \mathscr{F}_t]\mathbb{1}_A = E_Q[H\mathbb{1}_A \mid \mathscr{F}_t]$$

which finishes the proof. □

The next Proposition shows that, for every claim H, the non-linear expectation

$$\bar{\mathcal{E}}(H) = \operatorname{esssup}\{E_Q[H \mid \mathscr{F}_t] : Q \in \mathscr{M}_e\}$$

can be computed by considering a subset of $\mathscr{M}_e$ only: one can restrict to the set of martingale measure $\mathscr{M}_e^H(P)$ under which H is integrable. In particular, for every claim H, the non-linear expectation $\bar{\mathcal{E}}(H)$ agrees with the upper bound of the no-arbitrage interval $\pi_t^{\sup}(H)$. This links the superhedging-duality Proposition 55 with the pricing in financial markets.

Proposition 64. *For every $H \in L^0_+(\mathscr{F}_T)$ we have the equalities*

$$\operatorname{esssup}\{E_Q[H \mid \mathscr{F}_t] : Q \in \mathscr{M}_e\} = \operatorname{esssup}\{E_Q[H \mid \mathscr{F}_t] : Q \in \mathscr{M}_e^H\}$$

and

$$\operatorname{ess\,inf}\{E_Q[H \mid \mathscr{F}_t] : Q \in \mathscr{M}_e\} = \operatorname{ess\,inf}\{E_Q[H \mid \mathscr{F}_t] : Q \in \mathscr{M}_e^H\}$$

Proof. We only show the first equality, the other one is left as exercise. Using Lemma 63 and Lemma 60, it suffices to show the following: if there exits $Q \in \mathscr{M}_e$ with $E_Q[H \mid \mathscr{F}_t] = +\infty$, then $\pi_t^{\text{sup}}(H) = +\infty$.

In this regard, consider $Q \in \mathscr{M}_e$ with $E_Q[H \mid \mathscr{F}_t] = +\infty$ and some $Y_t \in L^0_t$. Then, there exists $n \in \mathbb{N}$ such that $\{Y_t \leq E_Q[H \wedge n \mid \mathscr{F}_t]\}$ has positive probability. By the fundamental theorem of asset pricing we find $\pi_t \in \Pi_t(H)$ such that $\{Y_t \leq \pi_t\}$ has positive probability.

Since Y_t was arbitrary, it follows that $\pi_t^{\text{sup}} = +\infty$ with positive probability. Now set $A := \{\pi_t^{\text{sup}} < +\infty\}$. Using Lemma 63, and arguing as above for the claim $H\mathbb{1}_A$, we deduce that $P(A) = 0$. $\square$

For the next lemma, recall that the smallest superhedging price $\mathcal{E}_t$ was defined in (46).

Lemma 65. *Consider the claim $C_T \in L^0_+(\mathscr{F}_T)$. Then, C_T is symmetric w.r.t. $\mathcal{E}_t$ if and only if C_T is attainable at time t.*

Proof. We start with some observations. Symmetry requires to consider $\mathcal{E}_t^*(\cdot) = -\mathcal{E}_t(-\cdot)$. This is the smallest subhedging price, and as a consequence of the superhedging-duality, Corollary 53,

$$\mathcal{E}_t^*(H) = \operatorname{esssup}\{C_t \in L^0_t : \exists\, \underline{H} \in \text{Pred} : H_t + G_t(\underline{H}) \leq C_T\}$$

is the largest sub-hedging price. Due to Lemma 50, $\mathcal{E}_t^*(H)$ is itself a sub-hedging price.

Now, suppose that C_T is attainable, i.e. $C_T = C_t + G_t(H)$ for some predictable process $H \in \text{Pred}$. Then,

$$\begin{aligned} \mathcal{E}_t^*(C_T) &= -\mathcal{E}_t(-C_T) = -\mathcal{E}_t(-C_t - G_t(H)) \\ &= \mathcal{E}_t^*(C_t) = C_t = \mathcal{E}_t(C_t + G_t(H)) = \mathcal{E}_t(C_T). \end{aligned}$$

On the contrary, if C_T is symmetric, $\mathcal{E}_t(C_T) = \mathcal{E}_t^*(C_T)$ is by definition finite. Hence there is a super- and a subhedging strategy such that

$$\mathcal{E}_t^*(C_T) + G_t(\underline{H}) \leq C_T \leq \mathcal{E}_t(C_T) + G_t(H)\,. \tag{66}$$

This implies

$$0 \leq H - \mathcal{E}_t(C_T) - G_t(H) \leq G_t(\underline{H} - H)\,.$$

By no-arbitrage, $G_t(H) = G_t(\underline{H})$ and so $C_T = \mathcal{E}_t(H) + G_t(H)$. $\square$

Theorem 67 (2nd fundamental theorem)**.** *The market is complete at time t if and only if every European contingent claim $C_T \in L^0_+(\mathscr{F}_T)$ has a unique price at time t.*

Proof. Assume that the market is complete. Then, by Lemma 65 and the superhedging duality, Proposition 55, every contingent claim has a unique arbitrage-free price.

On the contrary, if a contingent claim has a unique arbitrage-free price, the superhedging duality implies that C_T is symmetric and hence the claim is attainable. $\square$

One can additionally show a number of things: for example completeness is equivalent to the pasting property $\mathscr{M}_e \subset \mathscr{M}_e \odot_t Q$ with some $Q \in \mathscr{M}_e$. Moreover, if the market

For details, we again refer to Niemann & Schmidt (2024).

Continuous-time Finance

We start with a gentle introduction to semimartingale theory, relying on the scriptum on stochastic processes from last semester.

Semimartingale theory

Let us first visit some important examples for semimartingales. We recall that a process is called càdlàg, if it is a process which has almost surely left limits and is almost surely continuous from the right (RCLL - right continuous with left limits).

We will consider a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, P)$ satisfying the usual conditions, i.e. the filtration is right-continuous and $\mathscr{F}$ is complete (subsets of null-sets are $\mathscr{F}_0$-measurable).

The Poisson process

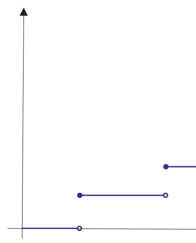

Figure 1: Path of a Po

Definition 68 (Poissonprocess). An adapted, càdlàg process X taking values in $\mathbb{N}$ is called *extended Poisson process*, if

(i) $X_0 = 0$,

(ii) $\Delta X_t \in \{0, 1\}$,

(iii) $E[X_t] < \infty$ for all $t \geq 0$,

(iv) $X_t - X_s$ is independent of $\mathscr{F}_s$, for $0 \leq s \leq t$.

We define the *cumulated intensity* Λ of X by

$$\Lambda(t) = E[X_t], \qquad t \geq 0.$$

Note that this is again an increasing, right-continuous process. X is called *Poisson-Prozess* with intensity $\lambda > 0$, if $\Lambda(t) = \lambda t,\ t \geq 0$.

If the cumulated intensity is absolutely continuous, i.e.

$$\Lambda(t) = \int_0^t \lambda(s)\, ds, \qquad t \geq 0$$

then the function λ is called the intensity of X. X is called standard Poissonprocess if $\lambda = 1$.

We note that we also may look at the time-transformed Poisson-process

$$X_{T_t}, \qquad t \geq 0,$$

Whenever T is increasing. If T is continuous and independent of X, we obtain that X is a Poisson process *conditional* on the filtration generated by the time-transformation T. If T has jumps we can no longer guarantee $\Delta X_t \in \{0, 1\}$.

The Poisson process corresponds one-to-one to a point process. Indeed if $T_n = \inf\{t \geq t : N_t \geq n\}$ denotes the n-th jumping time of N, then $(T_n)_{n\geq 1}$ are an increasing sequence of stopping time, so a *point process*.

If we additionally have a sequence $(Z_n)_{n\geq 1}$ of random variable on a Polish space E, then the double sequence $(T_n, Z_n)_{n\geq 1}$ constitutes a *marked point process*.

If we want to construct integrals over the marked point process we would be interested in expressions like

$$\sum_{n\geq 1} H(T_n, Z_n) = \int H(s,x)\mu(ds,dx)$$

where we can introduce the associated random measure

$$\mu(\omega; ds, dx) = \sum_{n\geq 1} \delta_{(T_n,Z_n)}(ds,dx),$$

where δ_a is the Dirac measure in point a.

Survival processes

In many applications, the first jump of the Poisson process is the most important one: mortality, default, insurance, etc. and it is therefore interesting to study this process in more generality.

Hence, consider a càdlàg process H with $H_0 = 0$ and a single jump of size 1. Then, this process is increasing, and hence by the Doob-Meyer decomposition there exists a unique compensator H^p which is a predictable process such that

$$H - H^p$$

is a local martingale. H^p takes over the role of a generalised intensity: indeed, in the Poisson example above, $H^p = \Lambda$. For a deeper study and applications to credit risk we refer to Gehmlich & Schmidt (2018).

Brownian motion

Definition 69. A continuous, adapted process W is called *Brownian motion* if

(i) $W_0 = 0$,

(ii) $E[W_t] = 0$ and $\mathrm{Var}(W_t) < \infty$, for all $t \geq 0$,

(iii) $W_t - W_s$ is independent of $\mathscr{F}_s$.

One can show that

$$W_t - W_s \sim \mathcal{N}(0, t-s),$$

i.e. the increments are even normally distributed. If we choose a time-change T appropriately, we can even construct a Poisson process as time-changed Brownian motion.

Classes of stochastic processes

To a stochastic process X we associate the mappgin

$$\widehat{X} \colon \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}^d.$$

This allows us to consider the stochastic process as a simple random variable on the product space $\Omega \times \mathbb{R}_{\geq 0}$. In particular, measurability can be considered with respect to $\mathscr{F} \otimes \mathscr{B}(\mathbb{R}_{\geq 0})$, which however lacks the link to the filtration.

Definition 70. (i) X is called *progressivly measurable*, if for all $t \geq 0$ the mapping

$$\begin{aligned} \Omega \times [0,t] &\to \mathbb{R}^d \\ (\omega, s) &\mapsto X_s(\omega) \end{aligned}$$

is $\mathscr{F}_t \otimes \mathscr{B}\big([0,t]\big)$-$\mathscr{B}(\mathbb{R}^d)$-measurable.

(ii) The *optional* σ-algebra $\mathscr{O}$ is the σ-algebra on $\Omega \times \mathbb{R}_{\geq 0}$, generated by adapted, càdlàg-processes. X is called *optional*, if $\widehat{X}$ is $\mathscr{O}$-measurable.

(iii) The *predictable* σ-algebra $\mathscr{P}$ is the σ-algebra on $\Omega \times \mathbb{R}_{\geq 0}$, generated by adapted, càg-processes. X is called *predictable*, if $\widehat{X}$ is $\mathscr{P}$-measurable.

In particular we obtain the following inclusions: predictable $\Rightarrow$ optional $\Rightarrow$ progressive $\Rightarrow$ adapted. A classical example is that for a progressive process X, $X^* = \sup_{s \leq \cdot} X_s$ is optional. Moreover, if we denote by X^T the process stopped at the stopping time T, then the following properties are kept while stopping: adapted, predictable, optional, progressive.

For the reverse consider an adapted process X. If X is càd, then X is progressive. If it is càg, then it is optional. If it is càdlàg, then X_- and $\Delta X = X - X_-$ are optional. For the following result, see the almost sure blog (see `https://almostsuremath.com/2016/11/15/optional-processes/`.)

Lemma 71. *Consider an adapted process X which is làd. Assume that X is càd everywhere except of a countable set $S \subset \mathbb{R}_{\geq 0}$. Then X is optional.*

We will often study random intervals, defined for two random times S and T by

$$[\![S, T]\!] := \{(\omega, t) \in \Omega \times \mathbb{R}_{\geq 0} : S(\omega) \leq t \leq T(\omega)\}.$$

As above we can call the interval optional or predictable if it is $\mathscr{O}$ resp. $\mathscr{P}$-measurable.

Localization

If we have a property $\mathscr{C}$ of a class of properties, then we introduce the localised class $\mathscr{C}_{loc}$ by all those processes X for which it holds there exists a sequence of stopping times $T_n \to \infty$ such that $X^{T_n} \in \mathscr{C}$ for all n. The sequence (T_n) is called localising sequence.

Definition 72. (i) A martingale X is uniformly integrable, if the family $(X_t)_{t \geq 0}$ is *uniformly integrable*. We denote by $\mathscr{M}$ the class of all uniformly integrable martingales.

(ii) A martingale X is called *square integrable*, if $\sup_{t \geq 0} E[X_t^2] < \infty$. This class is denoted by $\mathscr{H}^2$.

(iii) A process in $\mathscr{M}_{loc}$ is called *local martingale* and a process in $\mathscr{H}^2_{loc}$ *locally square integrable*.

Definition 73. A function $f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ has *locally finite variation*, if

$$\mathrm{Var}(f)_t := \sup_{0 \leq t_0 \dots t_n \leq t} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| < \infty$$

for all $t \geq 0$. A process is called of locally finite variation, if it has paths of locally finite variation.

Let $\mathscr{V}^+ :=$ denotes the increasing càdlàg process A, with $A_0 = 0$,

$$\begin{aligned}
\mathscr{V} &:= \mathscr{V}^+ - \mathscr{V}^+, \\
\mathscr{A}^+ &:= \{A \in \mathscr{V}^+ \colon E[A_\infty] < \infty\}, \\
\mathscr{A} &:= \mathscr{A}^+ - \mathscr{A}^+ = \{A \in \mathscr{V} \colon E[\mathrm{Var}(A)_\infty] < \infty\}.
\end{aligned}$$

Then $\mathscr{V}$ is the set of all adapted processes of locally finite variation. For each $A \in \mathscr{V}$ we can associate $t \mapsto A_t(\omega)$ with a signed measure, denoted by $dA_t(\omega)$. Then we can define (pathwise) for optional processes H,

$$(H \cdot A)_t(\omega) = \begin{cases} \int_0^t H_s dA_s & \text{falls } \int_0^t |H_s| d\,\mathrm{Var}(A)_s < \infty \\ \infty & \text{sonst.} \end{cases}$$

We obtained the following theorem.

Theorem 74 (Integral of finite variation processes). *Let $A \in \mathscr{V}(\mathscr{V}^+)$ and $H \geq 0$ be optional, such that $B = H \cdot A < \infty$. Then $B \in \mathscr{V}(\mathscr{V}^+)$. If H and A are predictable, so is B.*

We also obtained the important result that the only predictable local martingale with finite variation is $M = 0$ (Recall that the Brownian motion is of course predictable).

Theorem 75 (Dual predictable projection). *Consider $A \in \mathscr{A}^+_{loc}$. Then there is a unique predictable process $A^p \in \mathscr{A}^+_{loc}$, satisfying one of the following, equivalent properties*

(i) $A - A^p \in \mathscr{M}_{loc}$,

(ii) $E[A^p_T] = E[A_T]$ *for all stopping times T,*

(iii) $E\big[(H \cdot A)_\infty)\big] = E\big[(H \cdot A^p)_\infty)\big]$ *for all predictable $H \geq 0$.*

One can also directly project on the predictable σ-algebra, but here we have a more versatile tool, the *dual* predictable projection. It helps us to generate local martingales, which is of course very importance to classify absence of arbitrage.

As an example, you might want to check for an extended Poisson process X, $X^p = \Lambda$.

Semimartingales

We can now define the set of all square integrable martingales by

$$\mathscr{H}^2 = \{M \in \mathscr{M} \colon \sup_{t \geq 0} E[X_t^2] < \infty\}$$

.

Proposition 76 (Predictable covariation). *Let $M, N \in \mathscr{H}^2_{loc}$. Then there exists a unique predictable process $\langle M, N\rangle \in \mathscr{V}$, s.t.*

$$MN - \langle M, N\rangle \in \mathscr{M}_{loc}\,.$$

If $M, N \in \mathscr{H}^2$, then $\langle M, N\rangle \in \mathscr{A}$ and $MN - \langle M, N\rangle \in \mathscr{M}$.

The process $\langle M, N\rangle$ is called predictable covariation and $\langle M\rangle = \langle M, M\rangle$ (predictable) quadratic variation.

Now you could show that a Wiener process with $\sigma^2(t) = \mathrm{Var}(W_t)$ is a continuous, square-integrable martingale with $\langle W\rangle = \sigma^2(t)$.

Mit $\mathscr{L}$ bezeichnen wir die Teilmenge von $\mathscr{M}_{loc}$ für die $M_0 = 0$ gilt

Definition 77. (i) If the process X can be decomposed as

$$X = X_0 + M + A \tag{78}$$

with $M \in \mathscr{L}$ and $A \in \mathscr{V}$, then X is called a *semimartingale*. By $\mathscr{S}$ we denote the space of semimartingales.

(ii) If A is predictable, the decomposition in (78) is unique and we call X special. The space of special semimartingales is denoted by $\mathscr{S}_p$.

If a semimartingale is continuous, it is special and M and A in its decomposition are continuous. If a semimartingale has bounded jumps, it is also special. So the issue of not being a special semimartingale arises from the large jumps.

We even can show a little bit more: for every semimartingale, there exists the decomposition

$$X = X_0 + X^c + M + A$$

with a continuous local martingale X^c and a purely discontinuous local martingale $M \in \mathscr{H}^2_{loc}$ and $A \in \mathscr{V}$.

The stochastic integral

We call H simple, if

$$H = Y\mathbb{1}_{[\![0]\!]} \quad \text{oder} \quad H = Y\mathbb{1}_{]\!]S,T]\!]}$$

with stopping times S und T and bounded, $\mathscr{F}_S$-measurable Y. These are the prototypes of simple processes, where it is clear how to integrate them. Indeed, let us define for simple H its stochastic integral $H \cdot X$ with respect to a stochastic process X by

$$(H \cdot X)_t := \begin{cases} 0 & \text{if } H = Y\mathbb{1}_{[\![0]\!]} \\ Y \cdot (X_{T\wedge t} - X_{S\wedge t}) & \text{otherwise.} \end{cases} \tag{79}$$

By $\mathscr{E}$ we denote the space of simple (elementary) processes.

Theorem 80 (The stochastic integral)**.** *Let X be a semimartingale. The mapping $H \mapsto H \cdot X$ has an extension from $\mathscr{E}$ to the space of locally bounded, predictable processes, such that*

(i) $H \cdot X$ is adapted and càdlàg,

(ii) $H \mapsto H \cdot X$ is linear,

(iii) if predictable (H^n) converge pointwise to H, and is $|H^n| \leq K$ for a locally bounded, predictable process K, then

$$(H^n \cdot X)_t \xrightarrow{P} (H \cdot X)_t \qquad \forall\, t > 0.$$

We obtained the following properties:

(i) $H \cdot X$ is again a semimartingale.

(ii) If X is a local martingale, so is $H \cdot X$.

(iii) If $X \in \mathscr{V}$, then $H \cdot X$ is the Lebesgue-Stieltjes integral.

(iv) $(H \cdot X)_0 = 0$ and $H \cdot (X - X_0) = H \cdot X$.

(v) $K \cdot (H \cdot X) = (KH) \cdot X$.

(vi) $\Delta(H \cdot X) = H \cdot \Delta X$.

(vii) Is T predictable and Y $\mathscr{F}_T$-messbar, then

$$(Y \mathbb{1}_{[\![T]\!]}) \cdot X = Y \cdot \Delta X_T \mathbb{1}_{[\![T,\infty[\![}$$

If X is even locally square integrable we can allow a larger class of integrands.

Theorem 81. *Let $X \in \mathscr{H}^2_{loc}$. Then, $H \mapsto H \cdot X$ has an extension from $\mathscr{E}$ to L^2_{loc} such that*

(i) $H \cdot X \in \mathscr{H}^2_{loc}$

(ii) $H \in L^2(X) \iff H \cdot X \in \mathscr{H}^2$

(iii) For $X, Y \in \mathscr{H}^2_{loc}$ and predictable $K, M \in L^2_{loc}(X)$,

$$\langle H \cdot X, K \cdot Y \rangle = HK \cdot \langle X, Y \rangle\,.$$

For two semimartingales $X, Y \in \mathscr{S}$ we can define the quadratic covariation of X and Y by

$$[X, Y] = XY - X_0 Y_0 - X_- \cdot Y - Y_- \cdot X\,. \tag{82}$$

And we showed a number of properties: Consider $X, X' \in \mathscr{S}$ and $Y \in \mathscr{V}$. Then

(i) $[X, X'] \in \mathscr{V}$ and $[X] \in \mathscr{V}^+$,

(ii) $[X, Y] = \Delta X \cdot Y$,

(iii) if Y is predictable, then $[X, Y] = \Delta Y \cdot X$,

(iv) if either X or Y is continuous, then $[X, Y] = 0$.

(v) $[X, X']_t = \langle X, X' \rangle_t + \sum_{s \leq t} \Delta X_s \Delta X'_s$.

The Itô-formula

One major result was the following result on the semimartingale property of twice differentiable functions of semimartingales.

Theorem 83 (Itô-Formula)**.** *Consider a d-dimensional semimartingale* $X = (X^1, \dots, X^d)$ *and* $f \in \mathscr{C}^2(\mathbb{R}^d, \mathbb{R})$. *Then,* $f(X) \in \mathscr{S}$ *and*

$$\begin{aligned} f(X) = f(X_0) &+ \sum_{i \le d} D_i f(X_-) \cdot X^i \\ &+ \frac{1}{2} \sum_{i,j \le d} D_{ij} f(X_-) \cdot \langle X^{i,c}, X^{j,c} \rangle \\ &+ \sum_{0 \le s \le \cdot} \Big(f(X_s) - f(X_{s-}) - \sum_{i \le d} D_i f(X_{s-}) \Delta X_s^i \Big). \end{aligned} \tag{84}$$

As a first application we considered stochastic exponentials. Here Y was called a *sotchastic exponential*, if $X \in \mathscr{S}$ and

$$Y = 1 + Y_- \cdot X. \tag{85}$$

We denote the solution of (85) by $Y = \mathscr{E}(X)$.

As an important example we have met the geometric Brownian motion, $\mathscr{E}(W)$. If W is a standard Brownian motion, then

$$\mathscr{E}(W)_t = \exp\Big(W_t - \frac{1}{2} t\Big), \qquad t \ge 0.$$

Theorem 86. *Consier* $X \in \mathscr{S}$. *Then there exists a unique solution of (85) given by*

$$\mathscr{E}(X)_t := Y_t = \prod_{0 < s \le t} (1 + \Delta X_s) e^{-\Delta X_s} \cdot \exp\Big(X_t - X_0 - \frac{1}{2} \langle X^c \rangle_t\Big), \qquad t \ge 0.$$

Girsanovs theorem

We have already seen that measure changes are of prime importance in financial mathematics. The key tool here is Girsanovs theorem. Define for a stopping time T

$$P_T := P|_{\mathscr{F}_T}.$$

P' is called *locally absolutely continuous* w.r.t. P, if

$$P'_t \ll P_t, \qquad \forall t \ge 0;$$

which we denote by $P' \overset{loc}{\ll} P$. This is even equivalent to our localisation procedure (there exist stopping times $T_n \to \infty$ for which $P'_{T_n} \ll P_{T_n}, \qquad \forall n$.

Theorem 87. *Let* $P' \overset{loc}{\ll} P$. *Then there exists a unique P-martingale* $Z \ge 0$, *such that*

$$Z_t = \frac{dP'_t}{dP_t}, \quad t \ge 0. \tag{88}$$

Z is called density of P' w.r.t. P and $E[Z_t] = 1$ for all $t \geq 0$. If $P' \ll P$, then Z is uniformly integrable and

$$Z_\infty = \frac{dP'}{dP}.$$

A typical example is a geometric Brownian motion

$$Z_t = e^{aW_t - \frac{a^2 t}{2}}, \qquad t \geq 0,$$

which however is *not* uniformly integrable!

Theorem 89 (Girsanov)**.** *Let $P' \overset{loc}{\ll} P$ with density Z. Consider $M \in \mathscr{M}_{loc}(P)$ with $M_0 = 0$. Then*

$$M' = M - \frac{1}{Z} \cdot [M, Z]$$

is P'-almost surely well-defined and a P'-local martingale. If $[M, Z] \in \mathscr{A}_{loc}$, then

$$M'' = M - \frac{1}{Z_-} \langle M, Z \rangle$$

is a P'-local martingale.

As a typical application we consider

$$Z_t = \exp\Big(\theta W_t - \frac{\theta^2 t}{2}\Big), \qquad 0 \leq t \leq T,$$

hence $Z_t = Z_0 + Z_- \cdot \theta W_t$. Then

$$\begin{aligned} M'_t &= W_t - \frac{1}{Z} \cdot [W, Z]_t \\ &= W_t - \frac{1}{Z} \cdot \theta Z d\langle W \rangle_t = W_t - \theta t \end{aligned}$$

is a local martingale. Since quadratic variation is not changed by the measure change, $(W_t - \theta t)_{0 \leq t \leq T}$ is a Brownian motion under P'.

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