Homework accompanying the lecture „Basics in Applied Mathematics“

Homework 3

Hand in: Tuesday, 5.11.2024, after the lecture in the mailbox at the Math Institut
(Don't forget to put your name on your homework.
Please hand in your solutions in groups of two.)

Exercise 1 (4 points)

a) Let $X_1, \dots, X_n$ be random variables on some discrete probability space $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$ with existing expectations.

Prove that $\sum_{k=1}^n X_k$ also possesses an expectation and that the sum commutates with the expectation:

$$\mathbb{E}\left[\sum_{k=1}^n X_k\right] = \sum_{k=1}^n \mathbb{E}\left[X_k\right]$$

b) Prove that the expectation is monotone, meaning that for random variables X, Y on some discrete probability space $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$ with existing expectations and with $X(\omega) \leq Y(\omega)$ for all $\omega \in \Omega$, it holds

$$\mathbb{E}\left[X\right] \leq \mathbb{E}\left[Y\right].$$

Exercise 2 (6 points)

Let $X \sim B(n,p)$, meaning X is binomial distributed with parameters n and p. Analog let $Y \sim H(N,K,n)$ be hypergeometric distributed defined in Homework 1.4.

a) Evaluate $\mathbb{E}[X]$ and the variance of X, given by

$$\operatorname{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2.$$

b) Prove following equation using a proof by induction.

$$\sum_{j=0}^k \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}$$

c) Using item b) Evaluate $\mathbb{E}[Y]$.

d) We want to look at the limit for $K \to \infty$. Set $Y_K \sim H(2K, K, n)$ and $X \sim B(n, 0.5)$.

Prove that $\lim_{K\to\infty} \mathbb{P}(Y_K = k) = \mathbb{P}(X = k)$ for all $k = 1, \dots, n$.

Exercise 3 (6 points)

We look at the experiment of rolling a fair four sided dice. We denote the number of dots we have rolled by X. Afterwards we flip X coins and note the number of heads as Y. Let A_i be the event that we rolled the number i for $i \in \{1, 2, 3, 4\}$, meaning $X = i$. Further let B_j be the event that we tossed head j-times for $j \in \{1, 2, 3, 4\}$, meaning $Y = j$.

a) Evaluate the probability for all possible pairs of $A_i \cap B_j$ for $i, j \in \{1, 2, 3, 4\}$. Write these probabilities into a table.

b) Evaluate the probabilities of B_j using the law of total probability, i.e. determine the probability mass function of Y.

c) Give the probabilities of A_i given B_2, i.e. determine the probability mass function of Y given X.

d) Let C_1 be the event of observing tails exactly once. Calculate the conditional probability of C_1 given B_2.

Programming exercise 4 (2 points)

We want to simulate the expected first entree time of rolling a 6 with a fair dice. One possibile Algorithm is given below.

Algorithm 1 First entree time of rolling a 6

INPUT: $\emptyset$

OUTPUT: random time X with distribution $\mathbb{P}(X = x) = (\frac{5}{6})^{x-1}(\frac{1}{6})$.

1: success←FALSE; $x = 1$
2: **while** success==FALSE **do**
3: $Y \leftarrow$ uniform distributed digit on $1, \ldots, 6$
4: **if** $Y == 6$ **then**
5: $X \leftarrow x$
6: success← TRUE
7: **else**
8: $x = x + 1$
9: **end if**
10: **end while**
11: **return** X

Implement this algorithm. Embed a counter which counts the times of execution of the third line. Change the code such that we can get n independent draws at the same time.
Write your code into the homework03.py file. This can be executed using the terminal in the command line.