Homework accompanying the lecture „Basics in Applied Mathematics“

Homework 2

Hand in: Tuesday, 29.10.2024, after the lecture in the mailbox at the Math Institut
(Don't forget to put your name on your homework.
Please hand in your solutions in groups of two.)

Exercise 1 (4 points)

Four friends (let's call them A, B, C, D) always sit together in the back row of the school bus, which has 5 seats.

(a) How many distinct possibilities are there for the four friends to sit on the five seats?

(b) The bus-driver can't tell the difference between C and D. How many distinct possibilities does he observe for friends A, B, C, C on the five seats?

(c) Friends A and B have had a fight the previous day and get on the morning-bus at the same stop. They sit as far apart as possible on the back row. At the next stop friends C and D join them and leave the middle seat empty. How many seating combinations are possible in this szenario?

(d) In the evening all friends enter the bus simultaneously and randomly distribute themselves among the five seats of the back row. What is the probability that A and B sit directly next to each other?

HINT: Think of the fith seat as occupied by an imaginary friend E.

Exercise 2 (4 points)

Given a discrete probability space $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$. Prove or disprove (using a counterexample) following hypothesis:

For any sets $A, B, C \subseteq \Omega$ with $\mathbb{P}(A) > 0$ and $0 < \mathbb{P}(B) < 1$ holds:

(a) $\mathbb{P}(B|A) > \mathbb{P}(B)$ and $\mathbb{P}(C|B) > \mathbb{P}(C) \Rightarrow \mathbb{P}(C|A) > \mathbb{P}(C)$

(b) $\mathbb{P}(A|B) > \mathbb{P}(C)$ and $\mathbb{P}(A|B^c) > \mathbb{P}(C) \Rightarrow \mathbb{P}(A) > \mathbb{P}(C)$

(c) $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) \Leftrightarrow \mathbb{P}(A|B) = \mathbb{P}(A|B^c)$

Exercise 3 (4 points)

Let X be Poisson distributed with parameter 1, i.e. $\mathbb{P}(X = k) = \frac{1}{k!}e^{-1}$ for all $k \in \{0, 1, 2, ...\}$.

(a) Prove that the probability $\mathbb{P}(X \geq k)$ converges to zero for $k \to \infty$.

(b) Let $k_0 \in \mathbb{N}$. Prove that the conditional probability distribution

$$\mathbb{P}(X \in A | X \geq k_0) \quad \text{for } A \in \mathcal{P}(\mathbb{N}_{\geq k_0})$$

satisfies the axioms of Kolmogorov (refer definition 1.2).

(c) Prove that $\mathbb{P}(X = k | X \geq k)$ does not converge to zero for $k \to \infty$.

Exercise 4 (4 points)

We want to implement a function that recursively generates a random permutation of the set $\{1, ..., n\}$. The algorithm is described below. Prove that this algorithm follows a Laplace distribution on the space of permutations.

Hint: This can be proven using mathematical induction. Also note that the number of permutations of a set $\{1, ..., n\}$ is $n!$.

Algorithm 1 RAND.PERMUTATION

INPUT: n length of the permutation
OUTPUT: σ vector of length n defining a permutation

if $n == 1$ **then** $\sigma = (1)$
else
  $l \leftarrow$ uniformly distributed draw from the set $\{1, ...n\}$
  $\sigma' \leftarrow$ RAND.PERMUTATION$(n-1)$
  $\sigma = (\sigma'_1, ..., \sigma'_{l-1}, n, \sigma'_l, ..., \sigma'_{n-1})$
end if
return σ

Programming exercise 5 (2 points)

The main part of this exercise is to implement the algorithm RAND.PERMUTATION presented in exercise 4. Follow instructions of the jupyter notebook homework02.ipynb found in the git repository.