Homework accompanying the lecture „Basics in Applied Mathematics“

Homework 1

Hand in: Tuesday, 22.10.2024, after the lecture in the mailbox at the Math Institut
(Don't forget to put your name on your homework.
Please hand in your solutions in groups of two.)

Exercise 1 (4 points)

Let $n \geq 1$. A perfect dice is thrown n-times. Let $X_1, ..., X_n$ denote the resulting number of dots from each throw.

a) Define the state space Ω and the distribution $\mathbb{P}$ on Ω.

b) Define the following subsets Ω:
A : The first role shows an even number of dots.
B : Every throw yields an odd number of dots.
C : The sum over all throws is odd.
D : The first and the n-th throw show one dot.

c) Calculate the probabilities for all sets in b).

Exercise 2 (4 points)

Using the axioms of Kolmogorov defined in definition 1.2. prove (i)-(iii) from proposition 1.5, which are also listed below:

(i) For any set A holds $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$,

(ii) $A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$,

(iii) For any sequence of sets $(A_n)_{n\in\mathbb{N}}$ holds

$$\mathbb{P}\left(\bigcup_{n=0}^{\infty} A_n\right) \leq \sum_{n=0}^{\infty} \mathbb{P}(A_n).$$

Exercise 3 (4 points)

Let D_1 and D_2 be the number of dots for two independent throws of a perfect dice. For a function $f : \{1, ..., 6\}^2 \to \mathbb{R}$ we define the expected value $\mathrm{E}[f(D_1, D_2)]$ as

$$\mathrm{E}[f(D_1, D_2)] := \sum_{i,j=1}^{6} f(i,j)\mathbb{P}(D_1 = i, D_2 = j) = \frac{1}{6^2} \sum_{i,j=1}^{6} f(i,j) \ .$$

a) Calculate $\mathrm{E}[D_1]$.

b) Calculate $\mathrm{E}[D_1 + D_2]$

c) Let $f(i,j) = \begin{cases} i+j & \text{, if } i \neq j \\ 0 & \text{else.} \end{cases}$ Calculate $\mathrm{E}[f(D_1, D_2)]$.

d) At the Party organised by the mathematical student council the price of the beer is determined by roling two dice. A student gets a beer for free if he has a doublet ($D_1 = D_2$). Otherwise he has to pay the sum of dots of both throws times 30 cent. How much will a thirsty student have to pay on average for a beer?

Exercise 4 (4 points)

Prove that following functions are indeed probability mass functions, as defined in Def. 1.9.

a) (Bernoulli distribution) Let $n \in \mathbb{N}$ and $p \in [0,1]$. We define $f(k) := \binom{n}{k} p^k (1-p)^{n-k}$ and $\Omega = \{0, \dots, n\}$.

(Experiment: Denotes the probability that a coin, wich shows heads with probability p, tossed n-times shows heads exaktly k-times)

b) (Poisson distribution) Let $\lambda \in \mathbb{R}_{>0}$. For $n \in \mathbb{N}$ we define $f(n) := \frac{\lambda^n}{n!} e^{-\lambda}$ and $\Omega = \mathbb{N}$.

(Experiment: A Poisson Distribution models the radioactive decay of atoms, refer to Wiki page)

c) (Hypergeometric distribution) For $N, K, n \in \mathbb{N}$, where $K \leq N$ and $n \leq N$ we define $f(k) := \frac{\binom{K}{k} \cdot \binom{N-K}{n-k}}{\binom{N}{n}}$ and $\Omega = \{\max\{0, n-(N-K)\}, \dots, \min\{n, K\}\}$.

(Experiment: An Urn contains N balls with n of them marked. If K of the balls are pulled without replacement, then $f(k)$ denotes the probability that precisely k of the pulled balls are marked.)

Hint: For b) it helps to look at the polynomial $(x+1)^{a+b}$ and compare the coefficients using the binomial formula.

Programming exercise 5 (2 points)

Follow instructions of the jupyter notebook homework01.ipynb found in the git repository.