# Homework accompanying the lecture „Basics in Applied Mathematics“

## Homework 8

**Hand in:** Tuesday, 26.11.2024, after the lecture in the mailbox at the Math Institut
(Don't forget to put your name on your homework.
Please hand in your solutions in groups of two.)

### Exercise 1 (4 points)

For the points $x_0, \dots, x_n$ let $w(x) = \prod_{j=0}^{n}(x - x_j)$ be the interpolating polynomial and $L_i, i = 0, 1, \dots, n$ be the $i$-th Lagrange basis polynomial. Show that

$$L_i(x) = \frac{w(x)}{(x - x_i)w'(x_i)}.$$

### Exercise 2 (4 points)

Let

$$\omega_{n+1}(x) = \prod_{i=0}^{n}(x - x_i)$$

be the nodal polynomial for the points $(x_i)_{i=0}^n$. Show that its derivative is

$$\omega'_{n+1}(x_i) = \prod_{\substack{j=0 \\ j \neq i}}^{n}(x_i - x_j).$$

### Exercise 3 (4 points)

Show that for the fixed point iteration $x^{k+1} = \Phi(x^k)$ with the contraction $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ the error estimate

$$||x^k - x^*|| \leq \frac{q}{1-q}||x^k - k^{k-1}||$$

holds. To what extent is this estimate relevant for practical purposes?

### Programming exercise 4 (6 points)

Use the equivalent representations

$$x_i^{k+1} = a_{ii}^{-1}\left(b_i - \sum_{j \neq i} a_{ij}x_j^k\right), \quad x_i^{k+1} = a_{ii}^{-1}\left(b_i - \sum_{j<i} a_{ij}x_j^{k+1} - \sum_{j>i} a_{ij}x_j^k\right)$$

of the Jacobi and Gauss-Seidel methods to implement them. Test your programs for the linear equation system $Ax = b$ with

$$A = \begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ \vdots \end{bmatrix}$$

and the starting vector $x^0 = [1, 1, ...1]^T \in \mathbb{R}^n$ for $n = 10, 20, 40$. Stop the iteration when $\|x^k - x^{k+1}\|_2 \leq \delta$ with $\delta = 10^{-5}$. Comment on the dependence of the iteration numbers on the dimension $n$ of the system of equations.