Homework accompanying the lecture „Basics in Applied Mathematics“

Homework 7

Hand in: Tuesday, 03.12.2024, after the lecture in the mailbox at the Math Institut
(Don't forget to put your name on your homework.
Please hand in your solutions in groups of two.)

Exercise 1 (4 points)

a) Let $P \in \mathbb{R}^{n\times n}$ be the permutation matrix corresponding to the bijection $\pi : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$. Show that $P^T = P^{-1}$ and

$$P^{-1} = [e_{\pi^{-1}(1)}, \dots, e_{\pi^{-1}(n)}],$$

where $\{e_k\}$ are the canonical basis vectors.

b) Let $P \in \mathbb{R}^{n\times n}$ be a permutation matrix corresponding to the k-th and p-th Entry of a vector is swapped, where $p > k$ applies.

(i) Let $A \in \mathbb{R}^{n\times n}$. Determine PA and AP.

(ii) Now let $j < k$, $L = I_n - \ell_j e_j^T$ with the canonical basis vector $e_j \in \mathbb{R}^n$ and a vector $\ell_j = [0, \dots, 0, l_{j+1,j}, \dots, l_{n,j}]^T$. Show that a vector

$$\hat{\ell}_k = [0, \dots, 0, \hat{l}_{j+1,j}, \dots, \hat{l}_{n,j}]^T$$

exists such that with $\hat{L} = I_n - \hat{\ell}_j e_j^T$ the identity at $\hat{L} = PLP$ holds.

Exercise 2 (3 points)

Use the Gaussian elimination method *without* pivot search to solve the linear system of equations $Ax = b$ with

$$A = \begin{bmatrix} -1 & 16 & -4 & 3 \\ -3 & 20 & -22 & 0 \\ 1 & -16 & 1 & -2 \\ 3 & -6 & 4 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} -24 \\ -45 \\ 20 \\ 11 \end{bmatrix}$$

Also, determine the LU decomposition of A.

Exercise 3 (3 points)

Using appropriate Householder transformations, determine a QR decomposition for

$$A = \begin{bmatrix} 0 & 6 & 7 \\ 1 & 5 & -5 \\ 0 & 8 & 11 \end{bmatrix},$$

and use it to solve the equation $Ax = (1, 1, 1)^T$.

Programming exercise 4 (4 points)

Implement the Householder method for computing a QR decomposition. Use your program to solve the system of equations $Ax = b$ with the $n \times n$ Hilbert matrix A defined by $a_{ij} = (i + j - 1)^{-1}$, $1 \leq i, j \leq n$, and the right-hand side $b = [1, 2, \ldots, n]$ for $n = 3$ and $n = 10$.

Programming exercise 5 (4 points)

Consider the LU factorization of a tridiagonal matrix A and verify that L and U each have only two nontrivial diagonals: i.e. $L_{ij} = 0$ for $i > j + 1$ and $U_{ij} = 0$ for $j > i + 1$.
Use this insight to develop a much more efficient algorithm for the LU decomposition in this case, and test your method on the $(n \times n)$ example $A_{ii} = 4, A_{i,i+1} = A_{i-1,i} = -1$ for different n.