Recent developments in dependence modeling **2024**

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Exercise 1

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Submission in either german or english online via moritz.ritter@stochastik.uni-freiburg.de or mailbox 3.15 at the mathematical institute.To be able to solve the exercises, you need to read the script, which can be found on the homepage, up to page 13.

Exercise 1 (Fréchet-Hoeffding bounds inequality; 4 Points)**.** Let C be a 2-dimensional copula. Prove that for every (u,v) in $[0,1]^2$,

$$\max(u+v-1,0) \leq C(u,v) \leq \min(u,v).$$

Hint: Use the monotonicity in each component for the upper bound. For the lower bound use the inclusion–exclusion principle.

Exercise 2 (Survival Copula; 4 Points)**.** For a pair (X,Y) of real-valued random variables with joint distribution function H, the joint survival function is given by $\overline{H}(x,y) = P[X > x, Y > y]$. The marginal distribution functions of X and Y are denoted by F and G, respectively. The corresponding survival functions are given by $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, respectively. Let C be a copula for (X,Y), i.e., C is a copula such that $H(x,y) = C(F(x), G(y))$. Define the survival copula $\hat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$ and show that

$$\overline{H}(x,y) = \hat{C}(\overline{F}(x), \overline{G}(y)).$$

Moreover, prove that $\hat{C}$ is indeed a copula.

Exercise 3 (Invariance properties; 4 Points)**.** Let X and Y be random variables with distribution function F and G, respectively. Let F_{XY} be the joint distribution function of (X,Y) and C_{XY} be a copula for (X,Y), i.e., $F_{XY}(x,y) = C_{XY}(F_X(x), F_Y(y))$. If a and b are strictly increasing and continuous on $\mathrm{Ran}(X)$ and $\mathrm{Ran}(Y)$, respectively, then C_{XY} is also a copula for the random vector $(a(X), b(Y))$, i.e., the joint distribution function of $(a(X), b(Y))$ fulfills $F_{a(X)b(Y)}(x,y) = C_{XY}(F_{a(X)}(x), F_{b(Y)}(y))$. Thus, C_{XY} is invariant under strictly increasing transformations of X and Y.

Exercise 4 (Distributional transform; 4 Points)**.** Let X be a real-valued random variable with distribution function F, and let $V \sim \mathrm{U}(0,1)$ be an independent random variable uniformly distributed on $(0,1)$. The *generalized distribution function* of X is defined by

$$F(x,\lambda) = P(X < x) + \lambda P(X = x) \quad \text{or equivalently} \quad F(x,\lambda) = F(x-) + \lambda\big(F(x) - F(x-)\big),$$

and the *generalized distribution transform* U of X is given by $U = F(X,V)$.
Show that $U \sim \mathrm{U}(0,1)$.

Hint: Define $p = P\big(X < F^{-1}(\alpha)\big)$ *and* $q = P\big(X = F^{-1}(\alpha)\big)$ *and show the equality*

$$\{U \leq \alpha\} = \{X < F^{-1}(\alpha)\} \cup \{X = F^{-1}(\alpha),\, p + Vq \leq \alpha\}.$$

You can use without proof that $F(F^{-1}(\alpha)) \leq \alpha$ *and if* F *is continuous at* $F^{-1}(\alpha)$ *that it holds that* $F(F^{-1}(\alpha)) = \alpha$.