Exercises for the lecture „Probability Theory I“

Sheet 11

Submission deadline: Thursday, 17.07.2025, until 10:15 o'clock in the mailbox in the math institute
(You may deliver the exercise solutions in pairs.)

Exercise 1 (4 points)

Assume $\int_{\mathbb{R}} |\Phi_X(t)|dt < \infty$ for the characteristic function Φ_X of a random variable X. Prove that $\mathbb{P}^X \ll \lambda$ with

$$\frac{d\mathbb{P}^X}{d\lambda} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \Phi_X(t) d\lambda(t).$$

Exercise 2 (4 points)

(a) Let X be a random variable with $\mathbb{E}[|X|^n] < \infty$. Prove that its characteristic function Φ_X is n-times continuously differentiable and for $k = 0, \dots, n$,

$$\Phi_X^{(k)}(t) = \mathbb{E}\left[(iX)^k e^{itX}\right].$$

Hint: The estimate $\left|\frac{e^{ihx}-1}{h}\right| \leq |x|$ can be helpful.

(b) Let X real-valued random variable with characteristic function Φ_X and $\mathbb{E}[X^2] < \infty$. For $\sigma > 0$ we assume

$$\lim_{t \searrow 0} \frac{\Phi_X(t) - 1}{t^2} = -\frac{\sigma^2}{2}.$$

Prove that $\mathbb{E}[X] = 0$ and $\mathbb{E}[X^2] = \sigma^2$.
Hint: Use part (a).

(c) Use part (b) to derive the expectation and variance of $X \sim \mathcal{N}(0, \sigma^2)$.

Exercise 3 (4 points)

(a) Prove that a family $\{\mathcal{N}(\mu_i, \sigma_i^2) : i \in I\}$ of Gaussian distributions is tight if and only if the family $(\mu_i, \sigma_i^2)_{i \in I}$ is bounded.

(b) Prove that every probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is the weak limit of a sequence of discrete probability measures.

(please turn over)

Exercise 4 (4 points)

Check with the help of characteristic functions if $X_n \xrightarrow{\mathcal{D}} X$ in the following cases:

(a) $Y_n \sim \text{Poi}(n)$, $X_n := \frac{Y_n - n}{\sqrt{n}}$ and $X \sim \mathcal{N}(0,1)$.

(b) $Y_n \sim \text{Geom}(p_n)$ with $np_n \xrightarrow[n\to\infty]{} \lambda > 0$, $X_n := \frac{Y_n}{n}$ and $X \sim \text{Exp}(\lambda)$.

(c) $X_n \sim \mathcal{N}(\mu_n, \sigma_n^2)$ where $\mu_n \xrightarrow[n\to\infty]{} \mu$, $\sigma_n^2 \xrightarrow[n\to\infty]{} 0$ and $X \sim \delta_\mu$.

Exercise 5 (4 bonus points)

(a) Prove that $X_n \xrightarrow{\mathcal{D}} 0$ if and only if there exists $\delta > 0$ such that $\Phi_{X_n}(t) \to 1$ for $|t| \leq \delta$.

(b) Let $X_1, X_2, \dots$ be independent such that $S_n = \sum_{m=1}^n X_m$ converges in distribution. Prove that $(S_n)_{n\in\mathbb{N}}$ then also converges in probability.

Hint: Use part (a) to show that $S_n - S_m \to_{\mathbb{P}} 0$ for $m, n \to \infty$, i.e. for all $\varepsilon, \delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $\mathbb{P}(|S_n - S_m| > \varepsilon) < \delta$ for all $m, n \geq n_0$. Then deduce from this stochastic Cauchy criterion the existence of a limit in probability. Here, it is helpful to first prove that an almost sure convergent subsequence exists.

Exercises for self-monitoring

(1) Determine Φ_X for a uniformly distributed random variable $X \sim \mathcal{U}([a,b])$.

(2) Determine Φ_X for a binomial-distributed random variable $X \sim \text{Bin}(n,p)$.

(3) Recall the inversion formula for the characteristic function.

(4) Recall *Helly's selection theorem.*

(5) What is the relationship between weak convergence of probability measures and the convergence of the associated characteristic functions?