Exercises for the lecture „Probability Theory I“

Sheet 10

Submission deadline: Thursday, 10.07.2025, until 10:15 o'clock in the mailbox in the math institute
(You may deliver the exercise solutions in pairs.)

Exercise 1 (4 points)

Let $p, X_0 \in (0,1)$ and define for each $n \geq 1$ recursively

$$X_{n+1} = \begin{cases} 1 - p + pX_n & \text{with probability } X_n, \\ pX_n & \text{with probability } 1 - X_n. \end{cases}$$

Prove that $(X_n)_{n\in\mathbb{N}}$ is a martingale that converges a.s. and in L^1. Furthermore, determine the law of the limit $X_\infty = \lim_{n\to\infty} X_n$.
HINT: For determining the law, consider the process $(X_n(1-X_n))_{n\in\mathbb{N}}$.

Exercise 2 (4 points)

Let $X_1, \ldots, X_n, U_1, \ldots, U_n$ be independent random variables for which $X_1, \ldots, X_n$ are identically distributed with finite expectation and $U_1, \ldots, U_n$ are uniformly distributed on $(0, t)$. Denote by $U_{(1)} \leq U_{(2)} \leq \ldots \leq U_{(n)} \leq U_{(n+1)} =: t$ the order statistics of $U_1, \ldots, U_n$. Set $S_m := \sum_{k=1}^m X_k$ and $\mathcal{F}_m := \sigma(S_k, U_{(k+1)}; m \leq k \leq n)$.
Prove that $(S_m/U_{(m+1)})_m$ is a backwards martingale w.r.t. $(\mathcal{F}_m)_m$ and deduce with the optional sampling theorem that

$$\mathbb{P}\left(\bigcup_{m=1}^n \left\{ S_m/U_{(m+1)} \geq 1 \right\} \,\middle|\, S_n = y \right) \leq \min\left\{ \frac{y}{t}, 1 \right\}.$$

Exercise 3 (4 points)

Let $g : \mathbb{R} \to \mathbb{R}$ a measurable function and $U_g := \{x \in \mathbb{R} : g \text{ is discontinuous in } x\}$. Prove that $U_g \in \mathscr{B}(\mathbb{R})$.
HINT: Consider sets of the form

$$U_g^{\varepsilon,\delta}(y,z) := \{x \in \mathbb{R} : \exists y, z \in B_\delta(x) \text{ with } |g(y) - g(z)| > \varepsilon\},$$

show that these are Borel sets and construct U_g as a countable intersection and union of such sets.

(please turn over)

Exercise 4 (4 points)

(a) Let X, Y be independent random variables. Prove that $\Phi_{X+Y} = \Phi_X \Phi_Y$.

(b) Let X, Y be independent random variables, f_Y the Lebesgue density of Y and Φ_X the characteristic function of X. Prove that the characteristic function of the product XY is given by

$$\Phi_{XY}(t) = \int \Phi_X(ty) f_Y(y) dy.$$

(c) Prove rigorously that the characteristic function Φ_Z of a standard normal random variable $Z \sim \mathcal{N}(0,1)$ is given by

$$\Psi_Z(t) = e^{-\frac{1}{2}t^2}.$$

HINT: Derive a differential equation for $\Phi_Z(t)$ using partial integration.

(d) The *Laplace distribution* on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with parameter $\mu \in \mathbb{R}$ and $\sigma > 0$ has the Lebesgue density $f(x) = 1/(2\sigma)\exp(-|x-\mu|/\sigma)$. Let X be a Laplace distributed random variable with parameters μ and σ. Prove that its characteristic function is given by

$$\Phi_X(t) = e^{i\mu t}\frac{1}{1+\sigma^2 t^2}.$$

(e) Let X_1, X_2, X_3, X_4 be stochastically independent $\mathcal{N}(0,1)$-distributed random variables. Prove that $X_1X_2 - X_3X_4$ is Laplace distributed with parameters $\mu = 0$ and $\sigma = 1$.

Exercises for self-monitoring

(1) Under which assumptions do backwards martingales converge a.s. and in L^1?

(2) Deduce the strong law of large numbers from the convergence theorem for backwards martingales.

(3) Define *convergence in law.*

(4) Give an equivalent description of convergence in law using expectation.

(5) Does the sequence $(\delta_{1/n})_{n\in\mathbb{N}}$ of Dirac measures converge in law?

(6) Derive the characteristic function of the Dirac measure δ_a, $a \in \mathbb{R}$.