Exercises for the lecture „Probability Theory I“

Sheet 9

Submission deadline: Thursday, 03.07.2025, until 10:15 o'clock in the mailbox in the math institute
(You may deliver the exercise solutions in pairs.)

Exercise 1 (4 points)

A monkey randomly types a capital letter on a keyboard at any time $n \geq 1$, i.e. we observe a sequence $(U_n)_{n\in\mathbb{N}}$ of independent random variables that are uniformly distributed on $\{A, B, \dots, Z\}$. Let τ be the first time that the monkey has typed $ABRACADABRA$, i.e.

$$\tau = \inf\{n \geq 1 \mid U_{n-10} = A, U_{n-9} = B, \dots, U_n = A\}.$$

Determine $\mathbb{E}[\tau]$.

Hint: Consider a sequence of players, where at each time n a new player starts by betting one euro on $U_n = A$. In case he wins, he receives 26 euro, which he bets on $U_{n+1} = B$. In case of winning again, he receives 26^2 euro, places them on a bet on $U_{n+2} = R$, possibly wins 26^3 euro that are invested in a bet on $U_{n+3} = A$ and so on. If he loses one of the rounds, the game is over for him. Consider further the total payout to all active players for each point in time $n \in \mathbb{N}$.

Exercise 2 (4 points)

Let $(X_n)_{n\in\mathbb{N}}$ be a martingale satisfying $\sup_{n\geq 1} \mathbb{E}[X_n^2] < \infty$. Prove that X_n converges almost surely and in L^2 towards a random variable $X_\infty \in L^2(\mathbb{P})$.

Hint: For the existence of a limit in L^2, it is sufficient to prove that $(X_n)_{n\in\mathbb{N}}$ is a L^2-Cauchy sequence.

Exercise 3 (4 points)

(a) Let $\mathfrak{X}$ be a set of random variables for which an integrable majorant exists, i.e. there exists $Y \in L^1(\mathbb{P})$ with $|X| \leq Y$ for all $X \in \mathfrak{X}$. Prove that $\mathfrak{X}$ is uniformly integrable.

(b) Let $\mathfrak{X}$ be a set of random variables and assume existence of $p > 1$ such that $\mathbb{E}[|X|^p] < C < \infty$ for all $X \in \mathfrak{X}$. Prove that $\mathfrak{X}$ is uniformly integrable.

(c) Let U be uniformly distributed on $[0, 1]$. Prove or disprove if $\mathfrak{X} := \left\{ n\mathbb{1}_{[0,\frac{1}{n}]}(U) \mid n \in \mathbb{N} \right\}$ is uniformly integrable.

(d) Find an example of a uniformly integrable set $\mathfrak{X} = \{X_n \mid n \in \mathbb{N}\}$ of random variables that has no integrable majorant, i.e. such that $\mathbb{E}[\sup_{n\in\mathbb{N}} |X_n|] = \infty$.

(please turn over)

Exercise 4 (4 points)

(a) Assume $X_n \longrightarrow_{\mathbb{P}} X$ and $\mathbb{E}[|X_n|] \longrightarrow \mathbb{E}[|X|] < \infty$. Prove that $\{X_n : n \in \mathbb{N}\}$ is uniformly integrable (Proposition 5.28, (iii)⇒(i)).

(b) Let X be a random variable with $\mathbb{E}[|X|] < \infty$ and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ an increasing sequence of σ-algebras and define $\mathcal{F}_\infty = \sigma(\bigcup_{n\in\mathbb{N}} \mathcal{F}_n)$. Prove that for $n \to \infty$,

$$\mathbb{E}[X|\mathcal{F}_n] \longrightarrow \mathbb{E}[X|\mathcal{F}_\infty] \quad \text{a.s. and in } L^1.$$

Exercise 5 (4 bonus points)

Let $(X_n)_{n\in\mathbb{N}}$ be identically distributed and independent random variables satisfying

$$\mathbb{P}(X_1 = 1) = p = 1 - \mathbb{P}(X_1 = -1) = 1 - q$$

for $p \in (0,1) \setminus \{\frac{1}{2}\}$ and $a, b \in \mathbb{N}$. Let $S_0 = 0$, $S_n := \sum_{i=1}^n X_i$ and

$$\tau := \inf\{n \geq 0 \mid S_n \in \{-a, b\}\}.$$

Prove that

$$\mathbb{P}(S_\tau = -a) = \frac{1 - (p/q)^b}{1 - (p/q)^{a+b}}.$$

Hint: First, show that $((q/p)^{S_n})_{n\in\mathbb{N}}$ is a martingale with respect to a suitably chosen filtration. Then, use the optional sampling theorem.

Exercises for self-monitoring

(1) Recall the *martingale convergence theorem*. Pay close attention to its conditions. Does convergence in L^1 follow?

(2) Do non-negative supermartingales converge? What about non-positive submartingales?

(3) Recall the definition of *uniform integrability*.

(4) Give the definition of a branching process and explain its name.

(5) Under which condition does a martingale converge in L^1?

(6) Prove that $\mathbb{E}[|X_n - X|] \to 0$ implies $\mathbb{E}[|X_n|] \to \mathbb{E}[|X|]$.