Exercises for the lecture „Probability Theory I“

Sheet 8

Submission deadline: Thursday, 26.06.2025, until 10:15 o'clock in the mailbox in the math institute
(You may deliver the exercise solutions in pairs.)

Exercise 1 (4 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(\mathcal{A}_n)_{n\in\mathbb{N}}$ a sequence of independent σ-algebras.

(a) Let $\{I_k \mid k \in K\}$ be a partition of $\mathbb{N}$, i.e. $\bigcup_{k\in K} I_k = \mathbb{N}$ and $I_k \cap I_j = \emptyset$ for $j \neq k$. Prove that $(\sigma(\mathcal{A}_j : j \in I_k))_{k\in K}$ is an independent family.
HINT: Consider the sets $\mathcal{C}_k := \left\{\bigcap_{j\in J_k} A_j \mid J_k \subset I_k \text{ finite}, A_j \in \mathcal{A}_j\right\}$.

(b) We define the *terminal σ-algebra* $\mathcal{T}$ as

$$\mathcal{T} = \mathcal{T}(\mathcal{A}_1, \mathcal{A}_2, \dots) := \bigcap_{n\geq 1} \sigma\left(\bigcup_{m\geq n} \mathcal{A}_m\right).$$

Prove the 0-1-law of Kolmogorov: For every $A \in \mathcal{T}$ we have $\mathbb{P}(A) \in \{0, 1\}$.
HINT: Prove that $\mathcal{T}$ is independent of itself.

Exercise 2 (4 points)

Let $X_1, X_2, \dots$ be a sequence of real valued random variables defined on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$, $\mathcal{A}_n := \sigma(X_n)$ and $S_n := \sum_{k=1}^n X_k$. Prove or disprove with a counterexample whether or not the following events are part of the terminal σ-algebra $\mathcal{T}(\mathcal{A}_1, \mathcal{A}_2, \dots)$ (defined in Exercise 1):

(a) $\{\omega \in \Omega \mid X_n(\omega) = 0\}$ for fixed $n \in \mathbb{N}$,

(b) $\{\omega \in \Omega \mid X_n(\omega) = 0 \text{ for some } n \in \mathbb{N}\}$,

(c) $\{\omega \in \Omega \mid X_n(\omega) = 0 \text{ for finitely many } n \in \mathbb{N}\}$,

(d) $\{\omega \in \Omega \mid S_n(\omega) = 0 \text{ finitely many } n \in \mathbb{N}\}$,

(e) $\left\{\omega \in \Omega \;\middle|\; \limsup_{n\to\infty} S_n(\omega) > \liminf_{n\to\infty} S_n(\omega)\right\}$.

(please turn over)

Exercise 3 (4+2(bonus) points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space and $\mathcal{F} \subset \mathcal{A}$ a sub-σ-algebra. Prove the following results about conditional expectation (Proposition 4.12):

(a) For all $\varepsilon > 0$, we have $\mathbb{P}(|X| \geq \varepsilon | \mathcal{F}] \leq \varepsilon^{-2}\mathbb{E}[X^2|\mathcal{F}]$ $\mathbb{P}$-a.s.

(b) If $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[Y^2] < \infty$, we have
$$\mathbb{E}[XY|\mathcal{F}]^2 \leq \mathbb{E}[X^2|\mathcal{F}]\mathbb{E}[Y^2|\mathcal{F}] \quad \mathbb{P}\text{-a.s.}$$

(c) Let $\varphi : \mathbb{R}^k \to \mathbb{R}$ be convex with $\mathbb{E}[|\varphi(X)|] < \infty$. Then Jensen's inequality holds for conditional expectation, i.e.
$$\varphi(\mathbb{E}[X|\mathcal{F}]) \leq \mathbb{E}[\varphi(X)|\mathcal{F}] \quad \mathbb{P}\text{-a.s.}$$

HINT: We say that a function $\varphi : \mathbb{R}^k \to \mathbb{R}$ is convex if the *epigraph* $\{(x, y) \in \mathbb{R}^k \times \mathbb{R} : \varphi(x) \leq y\}$ is a convex set, where a set $C \subset \mathbb{R}^n$ is said to be convex if for all $x, y \in C$ and $0 \leq \lambda \leq 1$ we also have $\lambda x + (1 - \lambda)y \in C$. You can use the following fact without proof: For any convex set $C \subset \mathbb{R}^n$ and $p \in \partial C$, there exists $v_p \in \mathbb{R}^n$ such that $\langle v_p, p - x\rangle \leq 0$ for all $x \in C$. If you work out a solution for this, you get two additional points.

(d) Let $X_n \geq 0$, $X_n \nearrow X$ and $\mathbb{E}[X] < \infty$. Then, $\mathbb{E}[X_n|\mathcal{F}] \nearrow \mathbb{E}[X|\mathcal{F}]$ $\mathbb{P}$-a.s.
HINT: First, prove the convergence in $L^1(\mathbb{P})$.

Exercise 4 (4 points)

Let $(X_n)_{n\in\mathbb{N}}$ be a martingale and $p > 1$. Prove that
$$\mathbb{E}\left[\sup_{k\leq n} |X_k|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|X_n|^p\right].$$

HINT: Use Doob's maximal inequality and Hölder's inequality (Analysis III, also true for the special case of probability measures) to prove
$$\mathbb{E}\left[(|X|_n^* \wedge K)^p\right] = \mathbb{E}\left[\int_0^{|X|_n^*\wedge K} p\lambda^{p-1} d\lambda\right] \leq \frac{p}{p-1}\mathbb{E}\left[(|X|_n^* \wedge K)^p\right]^{\frac{p-1}{p}} \mathbb{E}\left[|X_n|^p\right]^{\frac{1}{p}}.$$

Then consider $K \to \infty$.

Exercises for self-monitoring

(1) Recall *Doob's maximal inequality.*

(2) Recall *Kolmogorov's inequality* and prove it.

(3) Recall the *upcrossing inequality.*

(4) Show $\mathbb{E}[Y_{\tau-1}] = -\infty$ for Y and τ given in Example 5.14 (doubling strategy).

(5) Let $(S_n)_{n\in\mathbb{N}}$ be the simple random walk given in Exercise 4 on Sheet 7. Prove that
$$\mathbb{P}\left(\sup_{k\leq n} |S_k| \geq \sqrt{n}t\right) \leq \frac{1}{t^2}.$$