# Exercises for the lecture „Probability Theory I“

## Sheet 4

**Submission deadline:** Wednesday, 18.06.2025, until 10:15 o'clock in the mailbox in the math institute
(You may deliver the exercise solutions in pairs.)

**Exercise 1** (4 points)

(a) Let $(X_n)_{n\in\mathbb{N}}$ a martingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ previsible with $|c_n| \leq k_n$ for all $n \in \mathbb{N}$. Prove that the martingale transform $(Y_n)_{n\in\mathbb{N}}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

(b) Let $(X_n)_{n\in\mathbb{N}}$ a sub-/supermartingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ previsible with $0 \leq c_n \leq k_n$ for all $n \in \mathbb{N}$. Prove that in this case the martingale transform $(Y_n)_{n\in\mathbb{N}}$ is a sub-/supermartingale w.r.t $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

**Exercise 2** (4 points)

Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and $(\mathcal{G}_n)_{n\in\mathbb{N}}$ filtrations with $\mathcal{F}_n \subseteq \mathcal{G}_n$, $n \in \mathbb{N}$ and $(X_n)_{n\in\mathbb{N}}$ a stochastic process adapted to both of them.

(a) Let $(X_n)_{n\in\mathbb{N}}$ be a martingale w.r.t. $(\mathcal{G}_n)_{n\in\mathbb{N}}$. Show that $(X_n)_{n\in\mathbb{N}}$ is also a martingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$.
In particular: If $(X_n)_{n\in\mathbb{N}}$ is a martingale w.r.t. $(\mathcal{G}_n)_{n\in\mathbb{N}}$, then $(X_n)_{n\in\mathbb{N}}$ is also a martingale w.r.t. the filtration induced by itself.

(b) Find an example of $(\mathcal{F}_n)_{n\in\mathbb{N}}$, $(\mathcal{G}_n)_{n\in\mathbb{N}}$ and $(X_n)_{n\in\mathbb{N}}$, such that $(X_n)_{n\in\mathbb{N}}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$ but not w.r.t. $(\mathcal{G}_n)_{n\in\mathbb{N}}$.

(c) Let $(\mathcal{H}_n)_{n\in\mathbb{N}}$ be another filtration such that $\mathcal{G}_n = \sigma(\mathcal{F}_n, \mathcal{H}_n)$ and $X_n$ is independent of $\mathcal{H}_m$ given $\mathcal{F}_m$ for all $n \geq m \geq 0$, i.e. $\mathbb{P}(X_n \in A|\sigma(\mathcal{F}_m, \mathcal{H}_m)) = \mathbb{P}(X_n \in A|\mathcal{F}_m)$. Prove the following: If $(X_n)_{n\in\mathbb{N}}$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$ then $(X_n)_{n\in\mathbb{N}}$ is a martingale w.r.t. $(\mathcal{G}_n)_{n\in\mathbb{N}}$ as well.

(d) Construct an example of a martingale $(X_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} X_n = \infty$ almost surely.

**Exercise 3** (4 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $(\mathcal{F}_n)_{n\in\mathbb{N}}$ a filtration and $\tau, \sigma$ stopping times with respect to this filtration.

(a) Prove that both $\tau \wedge \sigma$ and $\tau \vee \sigma$ are stopping times.

(b) Let $\tau, \sigma \geq 0$. Prove that $\tau + \sigma$ is a stopping time.

(please turn over)

Define the $\sigma$-algebra der $\tau$-past as

$$\mathcal{F}_\tau := \{A \in \mathcal{A} \mid A \cap \{\tau \leq n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}.$$

(c) Prove that $\tau$ is $\mathcal{F}_\tau$-measurable.

(c) Prove that $\mathcal{F}_\tau \cap \mathcal{F}_\sigma = \mathcal{F}_{\tau \wedge \sigma}$.

**Exercise 4** (4 points)

Let $(X_n)_{n\in\mathbb{N}}$ identically distributed and independent random variables with

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}.$$

We define $S_0 = 0$ and $S_n := \sum_{i=1}^n X_i$.

(a) Prove that $\mathbb{P}$-almost surely $\limsup_{n\to\infty} S_n = \infty$ and $\liminf_{n\to\infty} S_n = -\infty$.
HINT: Consider the sets $\{S_{n_k+k} - S_{n_k} = k\}$ for suitable $n_k$ and $k \in \mathbb{N}$ and use the theorem of Borel-Cantelli. Moreover, you can assume without proof that $\mathbb{P}(\limsup_n S_n \in A) \in \{0, 1\}$ for $A \in \mathscr{B}(\mathbb{R})$ (and the same for $\liminf_n S_n$). A proof of this is given in a subsequent exercise.

(b) Let $a, b \in \mathbb{N}$ and $\tau := \inf\{n \geq 0 \mid S_n \in \{-a, b\}\}$. Determine $\mathbb{P}(S_\tau = -a)$.
HINT: First, determine $\mathbb{E}[S_\tau]$.

**Exercises for self-monitoring**

(1) Define *filtration* and *(sub-/super-)martingale*.

(2) Let $(Y_n)_{n\in\mathbb{N}}$ independent random variables satisfying $\mathbb{E}[Y_n] = 1$ for all $n \in \mathbb{N}$ and $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$. Show that $(X_n)_{n\in\mathbb{N}}$ with $X_n = \prod_{k=1}^n Y_k$ is a martingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

(3) Let $(X_n)_{n\in\mathbb{N}}$ be a martingale w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Show that $\mathbb{E}[X_n] = \mathbb{E}[X_1]$ for all $n \in \mathbb{N}$.

(4) Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration and $m \in \mathbb{N}$. Prove that $\tau = m$ is a stopping time.

(5) Let $(X_n)_{n\in\mathbb{N}}$ be a martingale and $\tau$ a stopping time. Prove that $(X_{\tau\wedge n})_{n\in\mathbb{N}}$ is a martingale as well.

(6) Think heuristically about why martingales play a role in the description of fair games. What does (5) then tell you about fair games?

(7) State the *optional sampling theorem*.

(8) State *Wald's equation*.

(9) Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be martingales and $a, b \in \mathbb{R}$. Is $aX + bY$ then necessarily a martingale?

(10) Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be supermartingales. Is $Z := X \wedge Y = (\min(X_n, Y_n))_{n\in\mathbb{N}}$ then a sub- oder supermartingale?