Exercises for the lecture „Probability Theory I“

Sheet 6

Submission deadline: Thursday, 05.06.2025, until 10:15 o'clock in the mailbox in the math institute
(You may deliver the exercise solutions in pairs.)

Exercise 1 (4 points)

(a) Let X, Y be two independent, uniformly on $[0,1]$ distributed random variables. Show that

$$\mathbb{E}[X \,|\, \max(X,Y)] = \frac{3}{4}\max(X,Y).$$

(b) Generalize the assertion of part (a) to $\mathbb{E}[X_1 \,|\, \max(X_1,\dots,X_n)]$ for independent, uniformly on $[0,1]$ distributed random variables $X_1,\dots,X_n$ and prove this generalization.

Exercise 2 (4 points)

Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space and X a random variable that is exponentially distributed with parameter $\lambda > 0$. For $t > 0$ define $Y_t := \min\{X, t\}$. Prove that

$$\mathbb{E}[X|Y_t] = X\mathbb{1}_{\{X<t\}} + \left(t + \frac{1}{\lambda}\right)\mathbb{1}_{\{X\geq t\}}.$$

Hint: First, determine a family of sets that generates $\sigma(Y_t)$.

Exercise 3 (4 points)

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a measurable function and X, Y two real-valued random variables satisfying $\mathbb{E}[|f(X,Y)|] < \infty$.

(a) Prove that $\mathbb{E}[f(X,Y)|Y = y] = \mathbb{E}[f(X,y)]$ for X and Y being independent.

(b) Find a counterexample for (a) for dependent X and Y.

Now, let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent identically distributed and integrable random variables and N a $\mathbb{N}_0$-valued random variable that is independent of $(X_n)_{n\in\mathbb{N}}$ with $\mathbb{E}[N] < \infty$.

(c) Determine $\mathbb{E}\left[\sum_{i=1}^N X_i \,\middle|\, N = n\right]$.

(c) Prove that

$$\mathbb{E}\left[\sum_{i=1}^N X_i\right] = \mathbb{E}[N]\mathbb{E}[X_1].$$

(please turn over)

Exercise 4 (4 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(X_n)_{n\in\mathbb{N}}$ a sequence of random variables that converges to X almost surely. Prove that for every $\varepsilon > 0$ there exists a set $A \in \mathcal{A}$ with $\mathbb{P}(A^c) < \varepsilon$ such that the convergence is uniform on A, i.e.

$$\sup_{\omega\in A} |X_n(\omega) - X(\omega)| \overset{n\to\infty}{\longrightarrow} 0.$$

Exercises for self-monitoring

(1) Define the conditional expectation based on conditional probability and vice versa.

(2) List all properties of conditional expectation you are familiar with.

(3) Prove linearity of conditional expectation.

(4) Let $\mathcal{F} \subset \mathcal{G}$. Prove that $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{F}] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{F}]$ a.s.

(5) Prove that $\mathbb{E}[X^2|\mathcal{F}] \geq \mathbb{E}[X|\mathcal{F}]^2$ a.s.

(6) Formulate a monotone convergence result for conditional expectation.