Exercises for the lecture „Probability Theory I“

Sheet 5

Submission deadline: Friday, 30.05.2025, until 10:15 o'clock in the mailbox in the math institute
(You may deliver the exercise solutions in pairs.)

Exercise 1 (4 points)

(a) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables with $\mathbb{E}[X_n] = 0$ for all $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} \mathrm{Var}(X_n) < \infty$. Prove that with probability one the series $\sum_{n=1}^{\infty} X_n(\omega)$ converges.

HINT: Show that the sequence of partial sums is almost surely a Cauchy sequence. You can use the *Kolmogorov inequality* (a version of this will be proven later in the lecture): For an independent sequence $(X_n)_{n\in\mathbb{N}}$ with $\mathbb{E}[X_n] = 0$ and $\mathrm{Var}(X_n) < \infty$ for all $n \in \mathbb{N}$ we have

$$\mathbb{P}\left(\max_{1\le k\le n} |X_1 + \cdots + X_k| \ge x\right) \le \frac{\mathrm{Var}(X_1 + \cdots + X_n)}{x^2}.$$

(b) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables, let $A > 0$ and $Y_n := X_n \mathbb{1}_{\{|X_n|\le A\}}$ for $n \in \mathbb{N}$. Furthermore, assume that

(1) $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > A) < \infty$,
(2) $\sum_{n=1}^{\infty} \mathbb{E}[Y_n]$ converges,
(3) $\sum_{n=1}^{\infty} \mathrm{Var}(Y_n) < \infty$.

Prove that under the above assumptions $\sum_{n=1}^{\infty} X_n$ converges almost surely.

Exercise 2 (4 points)

(a) Let (X, Y) be a random vector with joint (Lebesgue-)density

$$f(x,y) = \frac{1}{8}\left(\mathbb{1}_{(-2,0)^2}(x,y) + \mathbb{1}_{[0,2)^2}(x,y)\right).$$

Determine the conditional densities $f_{X|Y=y}(x)$ and $f_{Y|X=x}(y)$ and the corresponding conditional expectations $\mathbb{E}[X|Y = y]$ and $\mathbb{E}[Y|X = x]$.

(b) Let $\mu = (\mu_X, \mu_Y)^T$ and $\Sigma = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}$. The vector (X, Y) is said to have a two-dimensional Gaussian distribution with expectation μ and covariance matrix Σ if its density with respect to Lebesgue measure is given as

$$f(x,y) = \frac{1}{2\pi\sqrt{\det(\Sigma)}} \exp\left(-\frac{1}{2}(x-\mu_X, y-\mu_Y)\Sigma^{-1}(x-\mu_X, y-\mu_Y)^T\right).$$

Determine the marginal density f_Y of Y, the conditional density $f_{X|Y=y}(x)$ and the conditional expectation $\mathbb{E}[X|Y = y]$.

(please turn over)

Exercise 3 (4 points)

Let N, $(X_n)_{n\in\mathbb{N}}$ be stochastically independent random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with $N(\Omega) \subset \mathbb{N}$ and $d\mathbb{P}^{X_n}/d\lambda = f_n$ for $n \in \mathbb{N}$ (i.e. f_n is a Lebesgue density of $\mathbb{P}^{X_n}$). Prove the following:

(a) $S_N := \sum_{i=1}^{N} X_i$ is measurable.

(b) For each $n \in \mathbb{N}$, the random variable $S_n = \sum_{i=1}^{n} X_i$ admits a density with respect to Lebesgue measure.

(c) Denote the density in (b) by $f_1 * \cdots * f_n$. Then, the function

$$g(x) := \sum_{n=1}^{\infty} \mathbb{P}(N = n)\,(f_1 * \cdots * f_n)\,(x)$$

is a Lebesgue-density of $\mathbb{P}^{S_N}$.

Exercise 4 (4 points)

Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \longrightarrow (\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0))$ be a random variable and define for $m \in \mathbb{N}_0$, $A \in \mathcal{A}$,

$$\widetilde{k}(m, A) := \begin{cases} \frac{\mathbb{P}(A\cap\{X=m\})}{\mathbb{P}(X=m)}, & \text{if } \mathbb{P}(X = m) > 0, \\ \mathbb{P}(A), & \text{if } \mathbb{P}(X = m) = 0. \end{cases}$$

For $\omega \in \Omega$ and $A \in \mathcal{A}$ we set $k(\omega, A) := \widetilde{k}(X(\omega), A)$. Prove the following:

(a) $\widetilde{k} : \mathbb{N}_0 \times \mathcal{A} \to [0, 1]$ is a Markov kernel from $(\mathbb{N}_0, \mathcal{P}(\mathbb{N}_0))$ to $(\Omega, \mathcal{A})$.

(b) $k : \Omega \times \mathcal{A} \to [0, 1]$ is a Markov kernel from $(\Omega, \sigma(X))$ to $(\Omega, \mathcal{A})$.

(c) For all $A \in \mathcal{A}$ and $C \in \sigma(X)$ we have $\mathbb{E}[\mathbb{1}_C k(\cdot, A)] = \mathbb{E}[\mathbb{1}_C \mathbb{1}_A]$.

Exercises for self-monitoring

(1) Define the *conditional probability* $\mathbb{P}(X = x|Y = y)$ for discrete random variables X and Y. Which problem do you encounter if Y is no discrete random variable?

(2) You throw a fair dice thrice and sum up all three results. What is the probability of an even sum given the first throw was an odd number?

(3) Define the *regular version of conditional probability* for X given $Y = y$, assuming X and Y to be real-valued random variables.

(4) Let X and Y be real-valued random variables and $B \in \mathscr{B}(\mathbb{R})$. Why does there exist a function $k(\cdot, B)$ such that

$$\mathbb{P}(X \in B, Y \in C) = \int_C k(y, B)d\mathbb{P}^Y(y)?$$

Is the left-hand side in the above equation a probability measure in C?