Exercises for the lecture „Probability Theory I“

Sheet 4

Submission deadline: Thursday, 22.05.2025, until 10:15 o'clock in the mailbox in the math institute
(You may deliver the exercise solutions in pairs.)

Exercise 1 (4 points)

Let $(X_n)_{n\geq 2}$ a sequence of independent random variables with

$$\mathbb{P}(X_n = -n) = \mathrm{P}(X_n = n) = \frac{1}{2n\log n} \qquad \text{and} \qquad \mathbb{P}(X_n = 0) = 1 - \frac{1}{n\log n}.$$

Moreover, let U uniformly distributed on $[0,1]$ and define for $n \geq 2$

$$Y_n := n \cdot \mathbb{1}_{\{U > 1 - \frac{1}{2n\log(n)}\}} - n \cdot \mathbb{1}_{\{U < \frac{1}{2n\log(n)}\}}.$$

Determine if for the sequence $(X_n)_{n\geq 2}$

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - \mathbb{E}[X_k]) \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^{n}(X_k - \mathbb{E}[X_k]) \to 0 \ a.s.$$

holds true and repeat this for $(Y_n)_{n\geq 2}$.
Hint: Show for the part with X_n that validity of the almost sure convergence would imply $\frac{X_n}{n} \to 0$ $a.s.$

Exercise 2 (4 points)

Let $M, K \in \mathbb{R}_{\geq 0}$ and $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued random variables such that $\mathrm{Var}(X_n) \leq M$ for all $n \in \mathbb{N}$ and X_i and X_j are uncorrelated whenever $|i-j| > K$. Prove that for $Z_n := \frac{1}{n}\sum_{i=1}^{n}(X_i - \mathbb{E}[X_i])$ we have $Z_{n^2} \to 0$ $\mathbb{P}$-almost surely.

Exercise 3 (4 points)

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. real-valued random variables with $\mathbb{E}[|X_1|] < \infty$ and $\mathcal{F}$ a class of measurable functions such that $f : \mathbb{R} \to \mathbb{R}$ and $\mathbb{E}[|f(X_1)|] < \infty$ for all $f \in \mathcal{F}$. Moreover, assume that for all $\varepsilon > 0$ there exists a finite set $\mathcal{G}_\varepsilon = \{h_1, \dots, h_N, g_1, \dots, g_N\} \subset \mathcal{F}$ such that

(i) For all $j \leq N$, $g_j \leq h_j$ pointwise and $\mathbb{E}[h_j(X_1) - g_j(X_1)] \leq \varepsilon$.

(ii) For every $f \in \mathcal{F}$ there exists $1 \leq j \leq N$ with $g_j \leq f \leq h_j$ pointwise.

Show that

$$\sup_{f\in\mathcal{F}} \left| \frac{1}{n}\sum_{i=1}^{n} f(X_i) - \mathbb{E}[f(X_1)] \right| \longrightarrow 0 \quad a.s.$$

(please turn over)

Exercise 4 (4 points)

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with $\mathbb{E}[|X_1|] < \infty$ and define

$$M_n = \frac{1}{n}\sum_{i=1}^{n} X_i X_{i+1}.$$

Show that the sequence $(M_n)_{n\in\mathbb{N}}$ converges almost surely and find its (almost sure) limit.

Exercises for self-monitoring

(1) State the *strong law of large numbers* of Etemadi.

(2) Recall the structure of the proof of strong law of large numbers.

(3) Define the *empirical distribution function* $\widehat{F}_n(x)$ and prove $\widehat{F}_n(x) \longrightarrow F(x)$ a.s.

(4) State the *Theorem of Glivenko-Cantelli*. Why is this not a direct consequence of the convergence in (2)?