Exercises for the lecture „Probability Theory I“

Sheet 3

Submission deadline: Thursday, 15.05.2025, until 10:15 o'clock in the mailbox in the math institute
(You may deliver the exercise solutions in pairs.)

Exercise 1 (4 points)

For $s \in (1, \infty)$, the Riemann zeta function is defined as the series

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}.$$

(a) Prove that $\mathbb{P}(\{n\}) := \zeta(s)^{-1} n^{-s}$ for $n \in \mathbb{N}$ defines a probability measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and that the family of events $\{x \in p\mathbb{N}\}$ for $p \in \mathcal{P} := \{p \in \mathbb{N} : p \text{ is a prime}\}$ is stochastically independent with respect to $\mathbb{P}$.

(b) Deduce from (a) that

$$\zeta(s) = \prod_{p \in \mathcal{P}} (1 - p^{-s})^{-1}.$$

Exercise 2 (4 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. For real-valued random variables X, Y we define

$$d(X, Y) := \inf\{\varepsilon > 0 : \mathbb{P}(|X - Y| > \varepsilon) < \varepsilon\}.$$

Prove the following:

(a) d is a pseudo-metric on the space of real-valued random variables (i.e. $d(X, Y) \in \mathbb{R}_{\geq 0}$ for all X, Y, d is symmetric, satisfies the triangle inequality and $d(X, X) = 0$ for all X).

(b) $d(X, Y) = 0$ if and only if $X = Y$ $\mathbb{P}$-almost surely.

(c) d induces stochastic convergence, i.e.

$$X_n \longrightarrow_{\mathbb{P}} X \qquad \Longleftrightarrow \qquad d(X_n, X) \xrightarrow{n \to \infty} 0$$

Exercise 3 (4 points)

(a) Let $(X_n)_{n \in \mathbb{N}}$ stochastically independent random variables satisfying

$$\mathbb{P}\left(X_n = \sqrt{n}\right) = \frac{1}{n} = 1 - \mathbb{P}\left(X_n = 0\right).$$

Determine if this sequence converges in probability, $\mathbb{P}$-almost surely or in L^p for any $p \geq 1$.

(please turn over)

(b) Prove the following stochastic Cauchy criterion: A sequence $(X_n)_{n\in\mathbb{N}}$ converges almost surely if and only if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcup_{m=1}^{\infty}\{|X_{m+n} - X_n| \geq \varepsilon\}\right) = 0.$$

Exercise 4 (4 points)

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent identically distributed random variables that are exponentially distributed with parameter $\alpha > 0$. Prove the following:

(a) $\mathbb{P}(\limsup_{n\to\infty} \frac{X_n}{\ln n} = \frac{1}{\alpha}) = 1$,

(b) $\mathbb{P}(\liminf_{n\to\infty} \frac{X_n}{\ln n} = 0) = 1$.

Hint: Use the Borel-Cantelli-Lemma. We have $\limsup_{n\to\infty} \frac{X_n}{\ln n} \leq \frac{1}{\alpha}$ if and only if for all $\varepsilon > 0$ only finitely many of the events $\{\frac{X_n}{\ln n} \geq \frac{1}{\alpha} + \varepsilon\}$ occur.

Exercises for self-monitoring

(1) Define all types of convergence for a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ that you are familiar with. Furthermore, list all implications between these convergence types.

(2) Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $X_1, Y_1, X_2, Y_2, \ldots$ random variables with $\mathbb{P}^{X_n} = \mathbb{P}^{Y_n}$ for alle $n \in \mathbb{N}$. Does $X_n \to_{\mathbb{P}} 0$ imply $Y_n \to_{\mathbb{P}} 0$?

(3) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables satisfying $X_n \to_{\mathbb{P}} X$ and $X_n \to_{\mathbb{P}} Y$. Show that $\mathbb{P}(X \neq Y) = 0$.

(4) Repeat part (3) for L^1-convergence instead of stochastic convergence.

(5) State the assertion of the *Borel-Cantelli-Lemma*.

(6) Express almost sure convergence using a criterion that is based on convergence in probability and vice versa.