Exercises for the lecture „Probability Theory I“

Sheet 2

Submission deadline: Thursday, 08.05.2025, until 10:15 o'clock in the mailbox in the math institute
(You may deliver the exercise solutions in pairs.)

Exercise 1 (4 points)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Prove the following:

(a) $\mathcal{A}^\mu := \{A \cup N : A \in \mathcal{A}, \text{ there exists } M \in \mathcal{A} \text{ with } \mu(M) = 0 \text{ and } N \subset M\}$ defines a σ-algebra.

(b) $\widetilde{\mu}(A \cup N) := \mu(A)$ for $A \cup N \in \mathcal{A}^\mu$ defines a measure $\widetilde{\mu}$ on $\mathcal{A}^\mu$.

(c) The measure space $(\Omega, \mathcal{A}^\mu, \widetilde{\mu})$ is *complete*, meaning that for all subsets $\widetilde{A}, \widetilde{B} \subset \Omega$ satisfying $\widetilde{A} \in \mathcal{A}^\mu$, $\widetilde{B} \subset \widetilde{A}$ and $\widetilde{\mu}(\widetilde{A}) = 0$ we automatically have $\widetilde{B} \in \mathcal{A}^\mu$.

Exercise 2 (4 points)

Let $(\Omega_i, \mathcal{A}_i) = ([0,1], \mathscr{B}([0,1]))$ for $i = 1, 2$ and $(\Omega, \mathcal{A}) = (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ the corresponding product space.

(a) Give an example of a set $A \subset \Omega$, such that for all $\omega_1 \in [0,1]$ we have for the ω_1-section $A_{\omega_1} := \{\omega_2 \mid (\omega_1, \omega_2) \in A\} \in \mathcal{A}_2$ and correspondingly $A_{\omega_2} \in \mathcal{A}_1$, but $A \notin \mathcal{A}$.
Hint: Recall (without proof) from your analysis course that $\mathscr{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$.

(b) Let $D = \{(x, x) \mid x \in [0,1]\}$ be the diagonal in Ω, λ the Lebesgue measure on Ω_1 and μ the counting measure on Ω_2, i.e.

$$\mu(A) = \begin{cases} |A|, & \text{if } A \text{ is finite,} \\ \infty & \text{else.} \end{cases}$$

Prove that $D \in \mathcal{A}$ and evaluate the integrals

$$\int_{\Omega_2} \int_{\Omega_1} \mathbb{1}_D(x, y) d\lambda(x) d\mu(y) \qquad \text{and} \qquad \int_{\Omega_1} \int_{\Omega_2} \mathbb{1}_D(x, y) d\mu(y) d\lambda(x).$$

(c) Is your result in (b) a contradiction to Fubini's theorem?

Exercise 3 (4 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and X a Weibull-distributed random variable, i.e. $\mathbb{P}^X$ admits the Lebesgue density $f_X(x) = \mathbb{1}_{(0,\infty)}(x) \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}$ with $\alpha, \beta > 0$. Find the distribution function $F_Y(t) := \mathbb{P}(Y \leq t)$ for $Y := \max\{X, 1\}$ and the Lebesgue decomposition of $\mathbb{P}^Y$ with respect to the Lebesgue measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

(please turn over)

Exercise 4 (4 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathcal{A}' := \{A \in \mathcal{A} \mid \mathbb{P}(A) \in \{0, 1\}\}$. Prove the following:

(a) $\mathcal{A}' \subset \mathcal{A}$ is a sub-σ-algebra,

(b) $\mathcal{A}$ and $\mathcal{A}'$ are stochastically independent.

(c) $\mathcal{A}'$ is stochastically independent of itself.

Exercises for self-monitoring

(1) Give the definition of the *product-σ-algebra*.

(2) When is a topological space called *Polish*? Recall a finite and an infinite dimensional example of such a space.

(3) Give the definition of a *(stochastic) kernel*.

(4) For which choice of the kernel do you recover Fubini's theorem from Theorem 1.29?

(5) Recall the definition of n σ-algebras being *independent*.

(6) Assume we have $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3)$ for events A_1, A_2 and A_3. Does this already imply independence of these events?