Exercises for the lecture „Probability Theory I“

Sheet 1

Submission deadline: Friday, 02.05.2025, until 10:15 o'clock in the mailbox in the math institute
(You may deliver the exercise solutions in pairs.)

Exercise 1 (4 points)

(a) Prove that the Borel σ-algebra $\mathscr{B}(\mathbb{R})$ on $\mathbb{R}$, which by definition is the σ-algebra generated by all open sets, is generated by the open intervals, i.e.

$$\mathscr{B}(\mathbb{R}) = \sigma(\{(a,b) \mid a, b \in \mathbb{R},\ a < b\}).$$

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that f is $\mathscr{B}(\mathbb{R}) - \mathscr{B}(\mathbb{R})$-measurable.

(c) For each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be a $\mathscr{B}(\overline{\mathbb{R}}) - \mathscr{B}(\overline{\mathbb{R}})$-measurable function. Prove that the pointwise defined functions $\sup_{n\in\mathbb{N}} f_n$ and $\limsup_{n\to\infty} f_n$ are again $\mathscr{B}(\overline{\mathbb{R}}) - \mathscr{B}(\overline{\mathbb{R}})$-measurable.

Exercise 2 (4 points)

(a) Let $\lambda > 0$ and μ be a measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ given by

$$\mu(\{n\}) = \frac{\lambda^n}{n!} e^{-\lambda} \quad \text{for all } n \in \mathbb{N}.$$

Based on this, define $\mu(A)$ for an arbitrary set $A \subset \mathbb{N}$ and explain why μ is a well-defined measure on $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Furthermore, calculate

$$\int_{\mathbb{N}} x \; d\mu(x).$$

(b) For $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) := n \mathbb{1}_{(0,\frac{1}{n}]}(x)$. Find the pointwise limit of $(f_n)_{n\in\mathbb{N}}$ and derive

$$\lim_{n\to\infty} \int_0^1 f_n(x) d\lambda(x).$$

Is this a contradiction to the theorem of dominated convergence?

Exercise 3 (4 points)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Moreover, $\mathbb{P}$ and $\mathbb{Q}$ are probability measures given by $\frac{d\mathbb{P}}{d\mu} = f$ and $\frac{d\mathbb{Q}}{d\mu} = g$. Prove that

$$\sup_{A\in\mathcal{A}} |\mathbb{P}(A) - \mathbb{Q}(A)| = \frac{1}{2} \int_\Omega |f - g| d\mu.$$

Hint: Consider the set $\{\omega \in \Omega : f(\omega) \geq g(\omega)\}$.

(please turn over)

Exercise 4 (4 points)

Let λ, μ and ν be measure on some measurable space $(\Omega, \mathcal{A})$. Prove:

(a) For $\lambda \ll \mu$ and $\mu \perp \nu$, we have $\lambda \perp \nu$.

(b) If $\mu \ll \nu$ and $\mu \perp \nu$ simulaneously, then $\mu \equiv 0$.

(c) If for all $\varepsilon > 0$ there exists $A \in \mathcal{A}$ such that $\mu(A) < \varepsilon$ and $\nu(A^c) < \varepsilon$, then $\mu \perp \nu$.

HINT: The required terms will be introduced on Monday.

Exercises for self-monitoring

(1) Recall the definition of the pushforward measure and show that it is actually a measure.

(2) Let μ be a probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ and F be given as

$$F(x) := \mu((-\infty, x]).$$

Argue that F is non-decreasing and right-continuous.

(3) Let F be a non-negative, non-decreasing and right-continuous function as in (2). For $a \leq b$, we define

$$\mu((a, b]) = F(b) - F(a).$$

Show that μ is a measure that is uniquely defined via F.

(4) Formulate all convergence theorems that you know for the integral. Pay close attention to the respective conditions!

(5) Denote by λ the Lebesgue measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. Find examples for a probability measure which

(i) is singular with respect to λ.

(ii) is absolutely continuous with respect to λ, but the opposite does not hold.

(iii) is equivalent to λ.

(iv) has both an absolutely continuous and a singular part (w.r.t. λ).