Albert-Ludwigs-Universität Freiburg

Lecture notes

Probability Theory I

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Contents

1 A short repetition of Measure Theory

This chapter contains a brief summary (without proofs) of the measure theoretic foundations as already known from our Analysis III course.

1.1 σ-Algebras and measures

Definition 1.1. *Let $\Omega \neq \varnothing$. A system $\mathscr{A}$ of subsets of Ω is called σ-algebra if*

(i) $\Omega \in \mathscr{A}$,

(ii) $A \in \mathscr{A} \implies A^c \in \mathscr{A}$,

(iii) $A_n \in \mathscr{A}\ \forall\, n \in \mathbb{N} \implies \bigcup_{n\in\mathbb{N}} A_n \in \mathscr{A}$.

$\mathscr{A}$ is called algebra if only (iii') instead of (iii) is granted:

(iii')

$$A, B \in \mathscr{A} \implies A \cup B \in \mathscr{A}.$$

Lemma 1.2. *Let $\Omega \neq \varnothing$ be a set and $\mathscr{A}$ a σ-algebra over Ω. Then the following statements hold true:*

(i) $A_n \in \mathscr{A}\ \forall\, n \in \mathbb{N} \implies \bigcap_{n\in\mathbb{N}} A_n \in \mathscr{A}$

(ii) $A, B \in \mathscr{A} \implies A \setminus B \in \mathscr{A}$.

Remark. *The pair $(\Omega, \mathscr{A})$ with a σ-algebra $\mathscr{A}$ over Ω is is called measurable space.*

Arbitrary intersections of σ-algebras over Ω are σ-algebras over Ω again. For any systems $\mathcal{E}$ of subsets of Ω, there exists a smallest σ-algebra $\sigma(\mathcal{E})$ with $\mathcal{E} \subset \sigma(\mathcal{E})$, namely the intersection of all those which contain $\mathcal{E}$.

Definition 1.3. *If Ω is equipped with a topology, then the σ-algebra generated by the open subsets of Ω is called Borel-σ-algebra.*

Definition 1.4. *Let $(\Omega, \mathscr{A})$ be a measurable space. A map $\mu : \mathscr{A} \to [0, \infty]$ with $\mu(\varnothing) = 0$ is called measure if it is σ-additive, meaning that*

$$A_n \in \mathscr{A}\ \forall n \in \mathbb{N},\ A_i \cap A_j = \varnothing\ \forall i \neq j \implies \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n).$$

The triple $(\Omega, \mathscr{A}, \mu)$ is called measure space. If $\mu(\Omega) = 1$, then μ is named probability measure and $(\Omega, \mathscr{A}, \mu)$ correspondingly probability space. A measure μ is called finite if $\mu(\Omega) < \infty$. It is named σ-finite if there exist sets $\Omega_i \in \mathscr{A}\ \forall i \in \mathbb{N}$ such that $\mu(\Omega_i) < \infty$ $\forall i \in \mathbb{N}$ and $\Omega = \bigcup_{i\in\mathbb{N}} \Omega_i$.

Notation. *The following notations are occasionally used variants of "$A \cup B$", but also convey the information that the sets A and B are disjoint:*

$$A \mathbin{\dot{\cup}} B, \; A + B.$$

Lemma 1.5. *Let $(\Omega, \mathscr{A})$ be a measurable space, μ a measure on $(\Omega, \mathscr{A})$, $A, B, A_n \in \mathscr{A}$ $\forall\, n \in N$. Then the following statements are satisfied:*

(i) $A \subset B \implies \mu(A) + \mu(B \setminus A) = \mu(B)$. *In particular,* $A \subset B \implies \mu(A) \leq \mu(B)$ *(monotonicity).*

(ii) $\mu(A \cap B) + \mu(A \cup B) = \mu(A) + \mu(B)$. *In particular,* $\mu(A \cup B) \leq \mu(A) + \mu(B)$ *(subadditivity).*

(iii) *If* $A_n \subset A_{n+1}$ $\forall\, n \in \mathbb{N}$, *then the continuity from below holds:*

$$\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

(iv) *If* $A_{n+1} \subset A_n$ $\forall n \in \mathbb{N}$ *and* $\mu(A_1) < \infty$, *then the continuity from above holds:*

$$\mu\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

We now pose the question under which conditions an additive functional (uniquely) extends to a measure.

Theorem 1.6 (Carathéodory's existence and uniqueness theorem)**.** *Let $\mathscr{A}$ be a system of subsets of Ω with the following properties:*

(i) $\Omega \in \mathscr{A}$,

(ii) $A, B \in \mathscr{A} \implies B \setminus A$ *is a finite disjoint union of sets of* $\mathscr{A}$,

(iii) $A, B \in \mathscr{A} \implies A \cap B \in \mathscr{A}$.

[Such a system of subsets of Ω is called half ring.] Let $\mu : \mathscr{A} \to [0, \infty]$ be a function with the following properties:

(iv) $\mu(\varnothing) = 0$,

(v) $A, B \in \mathscr{A}$ *with* $A \cup B \in \mathscr{A}$ *and* $A \cap B = \varnothing \implies \mu(A \cup B) = \mu(A) + \mu(B)$ *(additivity),*

(vi) *There exist* $\Omega_i \in \mathscr{A}$ *with* $\mu(\Omega_i) < \infty$ *and* $\Omega_i \subset \Omega_{i+1}$ $\forall\, i \in \mathbb{N}$ *such that* $\bigcup_{i\in\mathbb{N}} \Omega_i = \Omega$.

(vii) $A, A_n \in \mathscr{A}\ \forall n \in \mathbb{N}$ *with* $A \subset \bigcup_{n\in\mathbb{N}} A_n$

$$\implies \mu(A) \leq \sum_{n\in\mathbb{N}} \mu(A_n).$$

Then there exists a unique extension of μ *to a measure on* $\sigma(\mathscr{A})$*.*

Remark. *The system of sets*

$$\{(a,b] \cap \mathbb{R} \mid a, b \in \mathbb{R} \cup \{-\infty, +\infty\}, a \leq b\}$$

satisfies the conditions (i)-(iii).

Remark. *A probability measure* μ *is uniquely described by its values on a* $\cap$*-stable generator* $\mathscr{E}$ *of the* σ*-algebra.*

Proof. As μ is normed (i.e. $\mu(\Omega) = 1$) and additive, these values provide all the values on

$$\mathscr{E}' = \left\{ \bigcap_{i=1}^{K} A_i \;\middle|\; A_i \text{ or } A_i^c \in \mathscr{E}\ \forall\, i \leq K, K \in \mathbb{N} \right\}.$$

The set $\mathscr{E}'$ satisfies (*i*)-(*iii*):

(*i*) We can assume that $\varnothing \in \mathscr{E}$, as the goal is to describe μ by its values on $\mathscr{E}$, and we already know that $\mu(\varnothing) = 0$. Using $K = 1$ and $A_1 = \Omega$ (which is valid, as $\Omega^c = \varnothing \in \mathscr{E}$), $\Omega \in \mathscr{E}'$ follows from the definition of $\mathscr{E}'$.

(*ii*) Let $A = \bigcap_{i=1}^{K} A_i, B = \bigcap_{j=1}^{L} B_j \in \mathscr{E}'$.

$$\implies B \setminus A = B \cap A^c = \left(\bigcap_{j=1}^{L} B_j\right) \cap \left(\bigcup_{i=1}^{K} A_i^c\right) = \bigcup_{i=1}^{K} \underbrace{\overbrace{\left(A_i^c \cap \bigcap_{j=1}^{L} B_j\right)}^{\text{finite intersection of sets for which the set or its complement are in } \mathscr{E}}}_{\in \mathscr{E}'}$$

(*iii*) Let $A = \bigcap_{i=1}^{K} A_i, B = \bigcap_{j=1}^{L} B_j \in \mathscr{E}'$. Then,

$$A \cap B = \bigcap_{i=1}^{K} \bigcap_{j=1}^{L} (A_i \cap B_j)$$

is a finite intersection of subsets of Ω such that for every set, the set itself or its complement is in $\mathscr{E}$; hence, $A \cap B \in \mathscr{E}'$.

Therefore, the claim follows from Theorem 1.6. □

1.2 Measurable maps and measure integral

Definition 1.7. *Let* $(\Omega, \mathscr{A}), (\Omega', \mathscr{A}')$ *be measurable spaces. A map* $f : \Omega \to \Omega'$ *is called* *measurable* *(or* $\mathscr{A}$*-*$\mathscr{A}'$*-measurable) if*

$$f^{-1}(A') \in \mathscr{A} \ \forall A' \in \mathscr{A}'$$

where $f^{-1}(A') := \{\omega \in \Omega \mid f(\omega) \in A'\}$.

Lemma 1.8. *Let* $(\Omega, \mathscr{A}), (\Omega', \mathscr{A}')$ *be measurable spaces and* $\mathscr{E} \subset \mathscr{A}'$ *with* $\sigma(\mathscr{E}) = \mathscr{A}'$ *(meaning* $\mathscr{E}$ *is a generator of* $\mathscr{A}'$*). Then:*

$$f^{-1}(E') \in \mathscr{A} \ \forall E' \in \mathscr{E}' \iff f^{-1}(A') \in \mathscr{A} \ \forall A' \in \mathscr{A}'$$

This means that it is sufficient to check measurability on a generator of the σ*-algebra.*

Notation. *For the sake of readability, we define* $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ *and*

$$\mathscr{B}(\overline{\mathbb{R}}) = \{B \subset \mathbb{R} \mid B \cap \mathbb{R} \in \mathscr{B}(\mathbb{R})\} = \{B \cup E \mid B \in \mathscr{B}(\mathbb{R}), E \subset \{-\infty, +\infty\}\}.$$

Proposition 1.9. *Let* $(\Omega, \mathscr{A})$ *be a measurable space and* $f_n : \Omega \to \overline{\mathbb{R}}$ *an* $\mathscr{A}$*-*$\mathscr{B}(\overline{\mathbb{R}})$*-measurable function* $\forall n \in \mathbb{N}$*. Then, the functions*

$$\inf_{n \in \mathbb{N}} f_n, \ \sup_{n \in \mathbb{N}} f_n, \ \liminf_{n \to \infty} f_n, \ \limsup_{n \to \infty} f_n$$

are also $\mathscr{A}$*-*$\mathscr{B}(\overline{\mathbb{R}})$*-measurable.*

Monotone limits play a crucial role for the construction of the measure integral.

Definition 1.10. *Let* $A \subset \Omega$*. The function*

$$\mathbb{1}_A : \Omega \to \{0, 1\}, \ \omega \mapsto \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

is called *indicator function**. If* $A_k \in \mathscr{A}$ *and* $c_k \in \mathbb{R} \ \forall k \in \{1, \ldots, n\}$ *for some* $n \in \mathbb{N}$*, then* $\sum_{k=1}^{n} c_k \mathbb{1}_{A_k}$ *is called* $\mathscr{A}$*-elementary function.*

Measure integral for elementary functions

Let f be an elementary function, i.e. $f = \sum_{k=1}^{n} c_k \mathbb{1}_{A_k}$ with measurable sets $A_k \subset \Omega$. Then the elementary integral is defined as

$$\int f \ \mathrm{d}\mu := \sum_{k=1}^{n} c_k \mu(A_k).$$

If there is an alternative representation $f = \sum_{k=1}^{n'} c'_k \mathbb{1}_{A'_k}$, one can easily prove that

$$\sum_{k=1}^{n} c_k \mu(A_k) = \sum_{k=1}^{n'} c'_k \mu(A'_k).$$

Hence, this integral is well-defined.

Lemma 1.11. *Let $(\Omega, \mathscr{A})$ be a measurable space and $f : \Omega \to \mathbb{R}$ an $\mathscr{A}$-$\mathscr{B}(\mathbb{R})$-measurable function with $f \geq 0$. Then the following statements hold true.*

(i) There exists a sequence $(f_n)_{n\in\mathbb{N}}$ of elementary functions with $f_n \leq f_{n+1}$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n = f$.

(ii) There exist sets $A_k \in \mathscr{A}$ $\forall n \in \mathbb{N}$ and a sequence $(c_k)_{k\in\mathbb{N}}$ of non-negative real numbers such that $f = \sum_{k\in\mathbb{N}} c_k \mathbb{1}_{A_k}$.

That is: Any measurable function $f \geq 0$ is the monotone limit of elementary functions $f_n \nearrow f$.

The crucial idea is to define the integral for general non-negative, measurable functions f by monotone approximation: For non-negative elementary functions $f_n \nearrow f$,

$$\int \underset{=\lim_{n\to\infty} f_n}{\underset{\uparrow}{f}} \,\mathrm{d}\mu := \lim_{n\to\infty} \int f_n \,\mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int f_n \,\mathrm{d}\mu$$

To ensure that this integral is well-defined, we have to verify that

$$\lim_{n\to\infty} \int f_n \,\mathrm{d}\mu = \lim_{n\to\infty} \int g_n \,\mathrm{d}\mu$$

if g_n is another sequence of elementary functions with $g_n \nearrow f$. This is the content of the next lemma.

Lemma 1.12. *Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and $f : \Omega \to \mathbb{R}$ a measurable, bounded function with $f \geq 0$. If*

$$f = \sum_{n=1}^{\infty} \alpha_n \mathbb{1}_{A_n} = \sum_{n=1}^{\infty} \beta_n \mathbb{1}_{B_n}$$

with constants $\alpha_n, \beta_n > 0$ $\forall n \in \mathbb{N}$, then

$$\sum_{n\in\mathbb{N}} \alpha_n \mu(A_n) = \sum_{n\in\mathbb{N}} \beta_n \mu(B_n).$$

We summarize the important findings and state formally the definition.

Definition 1.13. *Let $f \geq 0$ be a non-negative measurable function. By Lemma 1.11, f is the monotone limit of non-negative elementary functions $f_n = \sum_{k=1}^{n} c_k \mathbb{1}_{A_k}$, that is $f = \sum_{n\in\mathbb{N}} c_k \mathbb{1}_{A_k}$, and we define*

$$\int f \, \mathrm{d}\mu := \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \sum_{n\in\mathbb{N}} c_n \mu(A_n).$$

By Lemma 1.12, this integral is well-defined.

A measurable (not necessarily non-negative) function $f : \Omega \to \mathbb{R}$ is called (finitely) integrable if $f^+ := f \cdot \mathbb{1}_{\{f\geq 0\}}$ and $f^- := -f \cdot \mathbb{1}_{\{f<0\}}$ are integrable, i.e.

$$\int f^+ \, \mathrm{d}\mu < \infty \text{ and } \int f^- \, \mathrm{d}\mu < \infty.$$

This is the case if and only if $\int |f| \, \mathrm{d}\mu < \infty$ (which we call absolute integrability). In this case, we define

$$\int f \, \mathrm{d}\mu := \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu.$$

Lemma 1.14 (Properties of the integral)**.** (*i*) *The integral is linear on the finitely integrable functions, i.e.*

$$\int (\alpha f + \beta g) \, \mathrm{d}\mu = \alpha \int f \, \mathrm{d}\mu + \beta \int g \, \mathrm{d}\mu$$

for all finitely integrable functions f, g and $\alpha, \beta \in \mathbb{R}$.

(*ii*) *The integral is monotone, i.e.*

$$g \leq f \implies \int g \, \mathrm{d}\mu \leq \int f \, \mathrm{d}\mu$$

for all integrable functions f, g.

(*iii*) *Theorem of monotone convergence: If $(f_n)_{n\in\mathbb{N}}$ is a sequence of finitely integrable or non-negative measurable functions with $f_n \leq f_{n+1} \; \forall \, n \in \mathbb{N}$, then*

$$\int \lim_{n\to\infty} f_n \, \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu.$$

(*iv*) *Theorem of dominated convergence: If $(f_n)_{n\in\mathbb{N}}$ is a sequence of measurable functions with $|f_n| \leq f \; \forall n \in \mathbb{N}$ for some function f with $\int f \, \mathrm{d}\mu < \infty$, such that the pointwise limit $g := \lim_{n\to\infty} f_n$ exists, then*

$$\int g \, \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu.$$

(*v*) <u>*Fatou's Lemma:*</u> *If* f, f_n *are finitely integrable and* $f \leq f_n$ $\forall n \in \mathbb{N}$*, then the function* $\liminf_{n\to\infty} f_n$ *is finitely integrable, and*

$$\liminf_{n\to\infty} \int f_n \, \mathrm{d}\mu \geq \int \liminf_{n\to\infty} f_n \, \mathrm{d}\mu.$$

1.3 Product spaces

Definition 1.15. *Let* I *be an arbitrary index set and* $(\Omega_i)_{i\in I}$ *a family of sets. We define*

$$\Omega = \prod_{i\in I} \Omega_i := \left\{ \omega : I \to \bigcup_{i\in I} \Omega_i \;\middle|\; \omega(i) \in \Omega_i \ \forall i \in I \right\}.$$

Its elements ω *are sometimes written as* $\omega = (\omega_i)_{i\in I}$*. Let* $\pi_i : \Omega \to \Omega_i$*,* $\omega \mapsto \omega_i$ *be the* <u>*coordinate map*</u> *(i.e. the projection onto the* i*-th coordinate).*

Definition 1.16. *Let* $(\Omega_i, \mathscr{A}_i)$ *be a measurable space for every* $i \in I$*. The smallest* σ*-algebra over* Ω *with respect to which the function* π_i *is measurable* $\forall\, i \in I$ *is called the* <u>*product-*σ*-algebra*</u>*. This means that the product-*σ*-algebra is*

$$\bigotimes_{i\in I} \mathscr{A}_i := \sigma\left(\pi_i^{-1}(A_i) \mid A_i \in \mathscr{A}_i \ \forall\, i \in I\right).$$

Theorem 1.17. *Let* $(\Omega_i, \mathscr{A}_i, \mu_i)$ *be* σ*-finite measure spaces* $\forall i \in \{1, \ldots, n\}$ *for some* $n \in \mathbb{N}$*. Then there exists a unique* σ*-finite measure* $\bigotimes_{i=1}^n \mu_i$ *on the product space*

$$\left(\prod_{i=1}^n \Omega_i, \ \bigotimes_{i=1}^n \mathscr{A}_i \right)$$

with the property

$$\left(\bigotimes_{i=1}^n \mu_i \right) (A_1 \times \ldots \times A_n) = \prod_{i=1}^n \mu_i(A_i)$$

for all sets $A_i \in \mathscr{A}_i$ $\forall i \in \{1, \ldots, n\}$*. We call this measure the* <u>*product measure*</u>*.*

Example. *Let* $\Omega_1 = \mathbb{R} = \Omega_2$*,* $\mathscr{A}_1 = \mathscr{B}(\mathbb{R}) = \mathscr{A}_2$*,* $\mu_1 = \lambda = \mu_2$*. Then the product measure* λ^2 *satisfies* $\lambda^2([a_1, b_1] \times [a_2, b_2]) = \lambda([a_1, b_1]) \cdot \lambda([a_2, b_2])$*.*

One nice property of the integral with respect to product measures is that it can be iteratively boiled down to integrals over the marginals.

Theorem 1.18 (Fubini)**.** *Let* $(\Omega_1, \mathscr{A}_1, \mu_1)$ *and* $(\Omega_2, \mathscr{A}_2, \mu_2)$ *be* σ*-finite measure spaces and* $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *a* $(\mathscr{A}_1 \otimes \mathscr{A}_2)$*-*$\mathscr{B}(\mathbb{R})$*-measurable function. If* f *is non-negative or* *absolutely integrable, then the maps*

$$\Omega_1 \to \overline{\mathbb{R}}, \omega_1 \mapsto \int_{\Omega_2} f(\omega_1, \omega_2) \, \mathrm{d}\mu_2(\omega_2) \quad \textit{and} \quad \Omega_2 \to \overline{\mathbb{R}}, \omega_2 \mapsto \int_{\Omega_1} f(\omega_1, \omega_2) \, \mathrm{d}\mu_1(\omega_1)$$

are measurable, and

$$\begin{aligned} \int_\Omega f \, \mathrm{d}\mu &= \int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) \, \mathrm{d}\mu_2(\omega_2) \right) \, \mathrm{d}\mu_1(\omega_1) \\ &= \int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2) \, \mathrm{d}\mu_1(\omega_1) \right) \, \mathrm{d}\mu_2(\omega_2) \end{aligned}$$

Definition 1.19. *A topological space is called* <u>*Polish*</u>*, if there exists a complete metric inducing its topology and a countable base of the topology.*

Example. *A well-known example of a Polish space is the set of all continuous functions on the unit interval, $\mathcal{C}([0,1])$, equipped with the topology of uniform convergence:*

$$d_{\sup} : \mathcal{C}([0,1]) \times \mathcal{C}([0,1]) \to \mathbb{R}, \ (f, g) \mapsto ||f - g||_\infty = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

Proposition 1.20 (Ulam's lemma)**.** *Let (Ω, τ) be a Polish space and $\mathscr{B}(\Omega)$ the Borel-σ-algebra on this topological space. Let μ be a probability measure on $(\Omega, \mathscr{B}(\Omega))$. Let $E \in \mathscr{B}(\Omega)$ and $\varepsilon > 0$. Then there exists a compact set $K \subset E$ with $\mu(E \setminus K) < \varepsilon$.*

Definition 1.21. *Let I be an index set, $(\Omega_i, \mathscr{A}_i)$ a measurable space for every $i \in I$ and*

$$(\Omega, \mathscr{A}) := \left(\prod_{i \in I} \Omega_i, \ \bigotimes_{i \in I} \mathscr{A}_i \right).$$

A family of probability measures $\{\mu_J \mid J \subset I \text{ is finite}\}$ on

$$(\Omega_J, \mathscr{A}_J) := \left(\prod_{i \in J} \Omega_i, \ \bigotimes_{i \in J} \mathscr{A}_i \right)$$

is called <u>*projective family*</u> *if for every finite set $J \subset I$ and every $K \subset J$, we have*

$$\mu_K = \mu_J^{\pi_K^J},$$

where $\pi_K^J : \Omega_J \to \Omega_K$, $(\omega_i)_{i \in J} \mapsto (\omega_i)_{i \in K}$ is the coordinate projection from Ω_J to Ω_K, and $\mu_J^{\pi_K^J}$ is the image measure of μ_J under π_K^J, meaning $\mu_J^{\pi_K^J}(A) = \mu_J((\pi_K^J)^{-1}(A))$ $\forall A \in \mathscr{A}_K$.