1. **The discrete case**

   Consider a market model $\bar{S} = (S^0, S^1, \dots, S^d)$ in one period with $\Omega = \{\omega_1, \dots, \omega_n\}$, and a strictly positive reference measure $\mathbb{P}$. Let $\bar{X} := \bar{S}/S^0$. Suppose that the market is free of arbitrage, i.e., for

   $$\mathcal{K} := \{\xi \cdot (X_1 - X_0) \colon \xi \in \mathbb{R}^d\}$$

   we have $\mathcal{K} \cap L^0_+ = \{0\}$.

   (a) Show that $\mathcal{K}$ can be identified as a subspace of $\mathbb{R}^n$. (2)

   (b) Let $C := \{x \in L^0_+ \colon \mathbb{E}[x] = 1\}$. Show that C is nonempty, convex and compact. (2)

   (c) Conclude that there exists $y \in \mathcal{K}^\perp$ such that $y \cdot x > 0$ for every $x \in C$. Further, show that $y(\omega_k) > 0$ for every $k \in \{1, \dots, n\}$. (2)

   (d) Show that there exists an equivalent martingale measure for the market. (2)

   Points for Question 1: 8

2. **Absolutely continuous measures**

   For a financial market with reference measure $\mathbb{P}$, we denote by $\mathcal{M}_e(\mathbb{P})$ and $\mathcal{M}_a(\mathbb{P})$ the set of equivalent martingale measures and absolutely continuous martingale measures, respectively.

   (a) Construct a market with $\mathcal{M}_a(\mathbb{P}) \neq \emptyset$ that contains an arbitrage. (2)

   (b) Assume $\mathcal{M}_e(\mathbb{P}) \neq \emptyset$, and let $H \in L^\infty(\mathbb{P})$ be a bounded claim. Show that (2)

   $$\sup\{\mathbb{E}^{\mathbb{Q}}[H] : \mathbb{Q} \in \mathcal{M}_a(\mathbb{P})\} = \sup\{\mathbb{E}^{\mathbb{Q}}[H] : \mathbb{Q} \in \mathcal{M}_e(\mathbb{P})\}\,.$$

   Points for Question 2: 4

You can achieve a total of **12** points for this sheet.