1. **Randomized Bolzano–Weierstraß**
   Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For $n \in \mathbb{N}$, let $X_n : \Omega \to \mathbb{R}$ be a random variable. Further, consider a sequence $(x_n) \subseteq \mathbb{R}$ with $\alpha := \liminf |x_n| < \infty$.

   (a) Show, by means of an example, that the sequence $(x_n)$ is not necessarily bounded. (1)

   (b) Set $\sigma_1 := 1$, and define further (2)

   $$\sigma_m := \inf\{n > \sigma_{m-1} : |\,|x_n| - \alpha| \leq 1/m\}.$$

   Prove that the sequence $(x_{\sigma_m})$ is bounded.

   (c) Conclude that $(x_n)$ has a convergent subsequence. (1)

   (d) Construct an example such that $\liminf_n |X_n|$ is $\mathbb{P}$-a.s. finite, but there exists no deterministic convergent subsequence of $(X_n)$. (2)

   Points for Question 1: 6

2. **Change of numeraire**
   Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t=0,\dots,T})$ be a filtered probability space, that carries a $(d+1)$-dimensional price process $\bar{S} = (S^0, S)$. Let $\mathcal{M}_e$ be the set of equivalent martingale measures. Assume $\mathcal{F}_0 = \{\emptyset, \Omega\}$, and that $S_t^1 > 0$. We define another discounted price process $\bar{Y}$ by $\bar{Y}_t := \bar{S}/S_t^1$ and denote the corresponding set of equivalent martingale measures by $\tilde{\mathcal{M}}_e$.

   (a) Show the equivalence (2)

   $$\mathcal{M}_e \neq \emptyset \iff \tilde{\mathcal{M}}_e \neq \emptyset.$$

   (b) Show that $\tilde{\mathcal{M}}_e$ is the set of measures $\tilde{\mathbb{Q}}$ such that $d\tilde{\mathbb{Q}}/d\mathbb{Q} = \frac{S_T^1/S_T^0}{S_0^1/S_0^0}$ for some $\mathbb{Q} \in \mathcal{M}_e$. (2)

   (c) Suppose the market is free of arbitrage. Show that $\mathcal{M}_e \cap \tilde{\mathcal{M}}_e \neq \emptyset \iff S_T^1/S_T^0$ is almost surely constant. (2)

   Points for Question 2: 6

You can achieve a total of **12** points for this sheet.