Artificial Intelligence

Albert-Ludwigs-Universität Freiburg

Thorsten Schmidt
Abteilung für Mathematische Stochastik

www.stochastik.uni-freiburg.de
thorsten.schmidt@stochastik.uni-freiburg.de
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Literature (incomplete, but growing):

- I. Goodfellow, Y. Bengio und A. Courville (2016). **Deep Learning**. `http://www.deeplearningbook.org`. MIT Press
- D. Barber (2012). **Bayesian Reasoning and Machine Learning**. Cambridge University Press
- R. S. Sutton und A. G. Barto (1998). **Reinforcement Learning : An Introduction**. MIT Press
- G. James u. a. (2014). **An Introduction to Statistical Learning: With Applications in R**. Springer Publishing Company, Incorporated. ISBN: 1461471370, 9781461471370
- T. Hastie, R. Tibshirani und J. Friedman (2009). **The Elements of Statistical Learning**. Springer Series in Statistics. Springer New York Inc. URL: `https://statweb.stanford.edu/~tibs/ElemStatLearn/`
- K. P. Murphy (2012). **Machine Learning: A Probabilistic Perspective**. MIT Press

Our goal today

Backpropagation

Regularization

Forward propagation

- In a feedforward neural network to produce an output $\hat{y}$ from an input x information flows forward through the network
- This is called forward propagation
- During training, forward propagation produces a scalar cost $J(\theta)$

Forward propagation algorithm for a typical deep neural net

- Require: Network depth, l
- Require: $W^{(i)}$, $i \in \{1,...,l\}$, the weight matrices of the model
- Require: $b^{(i)}$, $i \in \{1,...,l\}$, the bias parameters of the model
- Require: x, the input to process
- Require: y, the target output
- set $h^{(0)} = x$
- for $k = 1,\ldots,l$ do:
  - $a^{(k)} = b^{(k)} + W^{(k)}h^{(k-1)}$
  - $h^{(k)} = f(a^{(k)})$
- at the end of the loop set:
- $\hat{y} = h^{(l)}$
- $J(\theta) = L(\hat{y},y) + \lambda\Omega(\theta)$,where θ is $(W^{(i)},b^{(i)})$ $i \in \{1,...,l\}$

Backpropagation

- The back-propagation algorithm allows the information from the cost to flow backwards through the network, in order to compute the gradient

- The term back-propagation is not the whole learning algorithm
- Back-propagation is only a method to compute the gradient
- Another algorithm, e.g. stochastic gradient descent, is used to perform learning using this gradient.

- Computing an analytical expression for the gradient is straightforward
- Numerically evaluating such an expression can be computationally expensive
- The back-propagation algorithm does so using a simple and inexpensive procedure, that relates to the chain rule.

$$
\begin{aligned}
&\frac{\partial z}{\partial w}\\
=&\frac{\partial z}{\partial y}\frac{\partial y}{\partial x}\frac{\partial x}{\partial w}\\
=&f'(y)f'(x)f'(w)\\
=&f'(f(f(w)))f'(f(w))f'(w)
\end{aligned}
$$

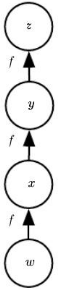

Figure from Goodfellow 2016

Backward propagation algorithm for a typical deep neural net

After the forward computation, compute the gradient on the output layer:

$\boldsymbol{g} \leftarrow \nabla_{\hat{\boldsymbol{y}}} J = \nabla_{\hat{\boldsymbol{y}}} L(\hat{\boldsymbol{y}}, y)$

for $k = l, l-1, \ldots, 1$ **do**

Convert the gradient on the layer's output into a gradient into the pre-nonlinearity activation (element-wise multiplication if f is element-wise):

$\boldsymbol{g} \leftarrow \nabla_{\boldsymbol{a}^{(k)}} J = \boldsymbol{g} \odot f'(\boldsymbol{a}^{(k)})$

Compute gradients on weights and biases (including the regularization term, where needed):

$\nabla_{\boldsymbol{b}^{(k)}} J = \boldsymbol{g} + \lambda \nabla_{\boldsymbol{b}^{(k)}} \Omega(\theta)$

$\nabla_{\boldsymbol{W}^{(k)}} J = \boldsymbol{g}\, \boldsymbol{h}^{(k-1)\top} + \lambda \nabla_{\boldsymbol{W}^{(k)}} \Omega(\theta)$

Propagate the gradients w.r.t. the next lower-level hidden layer's activations:

$\boldsymbol{g} \leftarrow \nabla_{\boldsymbol{h}^{(k-1)}} J = \boldsymbol{W}^{(k)\top}\, \boldsymbol{g}$

end for

this is Algorithm 6.4 in Goodfellow 2016

- The gradients on weights and biases can be used for a stochastic gradient update
- Symbol-to-number differentiation (Torch, Caffe): Use a set of numerical values for the inputs and return a set of numerical values describing the gradient at those input values
- Symbol-to-symbol differentiation (Theano,Tensorflow): Add additional nodes to the graph that provide a symbolic description of the desired derivatives.
- Because the derivatives are just another computational graph, it is possible to run back-propagation again, to obtain higher derivatives.

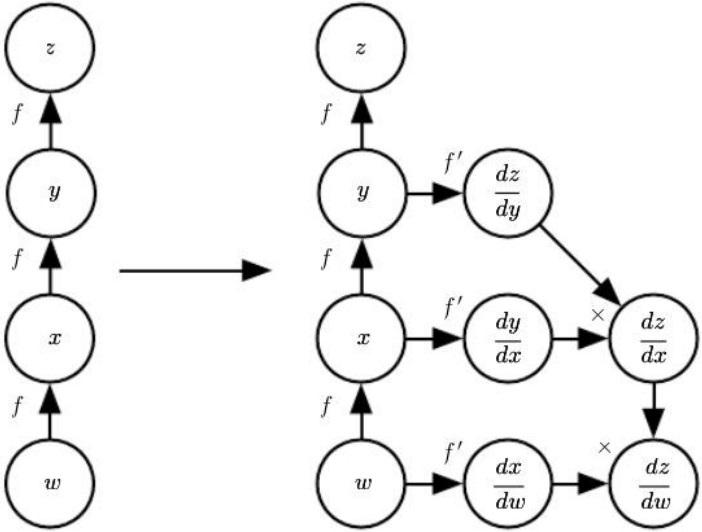

Figure from Goodfellow 2016

Regularization in Neural Networks

- regularization is a way to overcome underfitting, overfitting issues by trading variance of the prediction error against bias.
- $\mathbb{E}[L(\hat{y},y)] = \text{Irreducible Error} + \text{Bias}^2 + \text{Variance}$ (excercise)
- regularization is a modification to a learning algorithm that is intended to reduce its generalization error but not its training error.
- we have already seen bagging as a regularization method
- In the context of deep learning, most regularization strategies are based on regularizing estimators, by adding a parameter norm penalty $\Omega(\theta)$ to J

$$J(\theta;X,y) + \lambda\Omega(\theta)$$

weight decay

- weight decay refers to the L^2 penalty.
- also known as ridge regression
- if we do not punish the bias b the objective function for weight decay is given by

$$\tilde{J}(w;X,y) = \frac{\lambda}{2} w^T w + J(w;X,y)$$

- this means in a single gradient update step the update changes to

$$w \leftarrow (1 - \varepsilon\lambda) w - \varepsilon \nabla_w J(w;X,y)$$

- the addition of the weight decay term has modified the learning rule to shrink the weight vector on each step

- we make a quadratic approximation to the objective function in the neighborhood of the value w^*, the optimal weights where unregularized training cost is minimal

$$\hat{J}(w) = J(w^*) + \frac{1}{2}(w - w*)^T H(w - w^*)$$

- where H is the Hessian matrix of J with respect to w evaluated at w^*
- the minimum of the regularized version of $\tilde{J}$ is at

$$\tilde{w} = (H + \lambda I)^{-1} H w^*$$

- If we decompose $H = Q\Lambda Q^T$ into a diagonal matrix Λ and an orthonormal basis of eigenvectors Q we get

$$\tilde{w} = Q(\Lambda + \lambda I)^{-1} \Lambda Q^T w^*$$

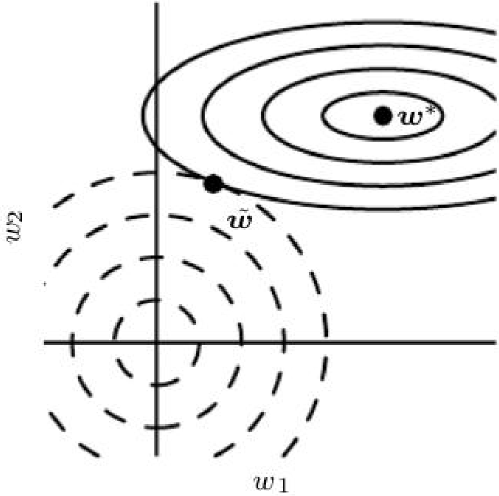

Figure from Goodfellow 2016

- In comparison to L^2 regularization, L^1 regularization results in a solution that is more sparse.
- Sparsity in this context refers to the fact that some weights have an optimal value of zero.

Let us have a look at the learning procedure at `playground.tensorflow.org`