Artificial Intelligence

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Our goal today

Dynamic Approximate Programming

Introduction

Markov decision problems

Approximate dynamic programming

Literature (incomplete, but growing):

- I. Goodfellow, Y. Bengio und A. Courville (2016). **Deep Learning**. http://www.deeplearningbook.org. MIT Press
- D. Barber (2012). **Bayesian Reasoning and Machine Learning**. Cambridge University Press
- R. S. Sutton und A. G. Barto (1998). **Reinforcement Learning : An Introduction**. MIT Press
- G. James u. a. (2014). **An Introduction to Statistical Learning: With Applications in R**. Springer Publishing Company, Incorporated. ISBN: 1461471370, 9781461471370
- T. Hastie, R. Tibshirani und J. Friedman (2009). **The Elements of Statistical Learning**. Springer Series in Statistics. Springer New York Inc. URL: https://statweb.stanford.edu/~tibs/ElemStatLearn/
- K. P. Murphy (2012). **Machine Learning: A Probabilistic Perspective**. MIT Press
- CRAN Task View: Machine Learning, available at https://cran.r-project.org/web/views/MachineLearning.html
- UCI ML Repository: http://archive.ics.uci.edu/ml/ (371 datasets)
- Warren B Powell (2011). **Approximate Dynamic Programming: Solving the curses of dimensionality**. Bd. 703. John Wiley & Sons

Dynamic Approximate Programming

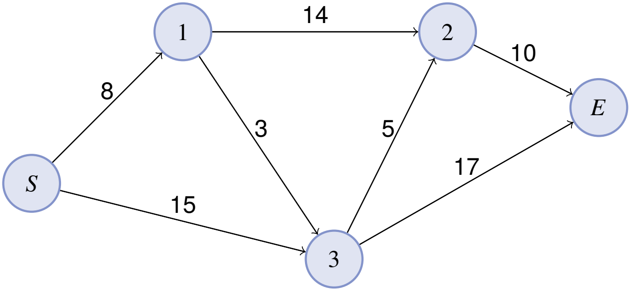
- From now on, we study the field of dynamic approximate programming (ADP) following Powell (2011)[1].
- As we already learned, there are many dialects in this field and we treat them here. This includes **reinforcment learning**, and a classic reference is Sutton & Barto[2]. For further references consider Powell (2011).
- Examples are: moving a robot, investing in stocks, playing chess or go.
- The system contains four main elements: a **policy**, a **reward funciton**, a **value function** and (optional) a **model** of the environment.

[1] Warren B Powell (2011). **Approximate Dynamic Programming: Solving the curses of dimensionality**. Bd. 703. John Wiley & Sons.

[2] R. S. Sutton und A. G. Barto (1998). **Reinforcement Learning : An Introduction**. MIT Press.

An Example

Let us start with a simple example.



It is our goal to find the shortest path from Start to End.

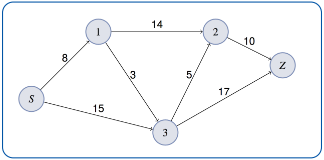
- By $\mathscr{I}$ we denote the set of intersections $(S,1,\ldots,E)$,
- if we are at intersection i we can go to $j \in \mathscr{I}_i^+$ at cost c_{ij},
- we start at S and end in E. Denote

$$v_i := \text{cost from } i \text{ to } E$$

and we could iterate

$$v_i \leftarrow \min\left\{v_i, \min_{j\in\mathscr{I}_i}(c_{ij}+v_j)\right\}, \quad v_i \in \mathscr{I}$$

and stop if the iteration does not change.



Iteration	S	1	2	3	E
1	∞	∞	∞	∞	0
2	∞	∞	10	15	0
3	30	18	10	15	0
4	26	18	10	15	0

- What is an efficient algorithm for solving this problem ?

- This is a **shortest-path problem**. Let us introduce some notation for this. At time t, we start from a state S_t and can choose **action** a_t which leads to the transition to state S_{t+1} given by the **transition function** S, s.t.

$$S_{t+1} = S(S_t, a_t)$$

- Aditionally there is a **reward**, denoted by $C_t(S_t, a_t)$ and we define the value of being in state S_t by

$$V_t(S_t) = \max_{a_t} \{C_t(S_t, a_t) + V_{t+1}(S_{t+1})\}, \qquad S_t \in \mathscr{S}_t,$$

  cS_t denoting the possible states at time t.
- Let us visit some further examples.

Gambling

- Consider a gambler who plays T rounds, on an i.i.d. $(W_t)_{t=1,\dots,T}$ game with probability $p = \mathbb{P}(W_t = 1) > 1 - p$ of winning. We want to maximize $\mathbb{E}[\log(S_T)]$. It can be shown that it is optimal to proceed backwards in time using conditional expectations (this is dynamic programming)!
- Here, a_t is the amount he bets at t and we require $a_t \le S_{t-1}$. Then,

$$S_t = S_{t-1} + a_t W_t - a_t(1 - W_t).$$

- The value at time t, given his stock is in state S_t is

$$V_t(S_t) = \max_{0 \le a_{t+1} \le S_t} \mathbb{E}\big[V_{t+1}(S_{t+1})|S_t\big].$$

- Now we proceed backwards. Clearly,

$$
\begin{aligned}
V_T(s) &= \log s \\
V_{T-1}(s) &= \max_{0\le a\le s} \mathbb{E}\big[V_T(s + aW_T - a(1-W_T))|S_{T-1} = s\big] \\
&= \max_{0\le a\le s} \Big(p\log(s+a) + (1-p)\log(s-a)\Big).
\end{aligned}
$$

- The maximum is attained for $a^* = (2p-1)s$ and $V_{T-1}(s) = \log(s) + K$, with costant $K = p\log(2p) + (1-p)\log(2(1-p))$. Backward in time we obtain

$$V_t(s) = \log S_t + K_t,$$

with an explicit constant K_t. Our **optimal policy** is

$$a_t = (2p-1)S_{t-1}.$$

The bandit problem

- When the distribution of the game is not known, one has to acquire information, and the classical example is the bandit problem. Consider a gambler who can choose betwee K machines.
- The probability of winning might be different and are **unknown** to us.
- A trade-off arises between playing only the optimal machine or trying other machines with (estimated) lower probability for minimizing the variance which is one-to-one to learning better their true probability.
- For a nite treatment, consider for example Richard Weber (1992). „On the Gittins Index for Multiarmed Bandits“. In: **Ann. Appl. Probab.** 2.4, S. 1024–1033.

Markov decision problems

- We give a short introduction into the field[3]. Assume that the state space $\mathscr{S}$ if **finite**.
- We have a set $\mathscr{A}_t(s)$ of possible actions at time t when the system is in state s. An action at t is a measurable mapping a_t such that $a_t(s) \in \mathscr{A}_t(s)$ for all $s \in \mathscr{S}$.
- A **policy** is a collection of actions $\pi = (a_0, \dots, a_{T-1})$. We assume that the set of policies is non-empty.
- The dynamics of the model is specified via the (conditional) transition matrix

$$(p_t(s_{t+1}|s_t, a_t))_{s_{t+1}, s_t \in \mathscr{S}}$$

  specifying $\mathbb{P}(S_{t+1} = s_{t+1}|S_t = s_t, a_t) = p_t(s_{t+1}|s_t, a_t)$.
- Hence, the dynamics and with it the probability for evaluation depends on π. We denote

$$\mathbb{P}^{\pi}_{t,s}(\cdot) := \mathbb{P}^{\pi}(\cdot|S_t = s)$$

  and by $\mathbb{E}^{\pi}_{t,s}$ the associated expectation.

[3]See N. Bäuerle und U. Rieder (2011). **Markov decision processes with applications to finance**. for details and further information.

- Our aim is to **maximize** the contribution given by the functions $C_t(s,a)$ where $C_T(s,a) = C_T(s)$ does not depend on a. We additionally assume that the contribution is sufficiently integrable.
- Our goal is to aim at

$$\sup_{\pi} \mathbb{E}^{\pi}\left[\sum_{0=1}^{T} C_t(S_t, a_t)\right].$$

For example, we could consider $C_t(s,a) = \gamma^t C(s,a)$ with possible discounting factor $\gamma > 0$.

The Bellman Equation

- The key to dynamic programming is that in our set-up, allowing the policy to depend on the full history does not improve the maximal expected reward, see Theorem 2.2.3. in Bäuerle&Rieder (2011).
- We define the **value function** by

$$V_t(s) = \sup_{\pi} \mathbb{E}^{\pi}_{t,s}\left[\sum_{s=t}^{T} C_t(S_t, a_t)\right].$$

Remark

In general V_t need not be measurable which causes a number of delicate problems, see D. P. Bertsekas und S. Shreve (2004). **Stochastic optimal control: the discrete-time case**. *for a detailed treatment. The reason can be traced back to the fact that a projection of a Borel set need not be Borel (which leads to the fruitful notion of analytic sets, however).*

- Define

$$C_t^*(s) := \sup_{a_t \in \mathscr{A}_t} \left(C_t(s,a_t) + \mathbb{E}\Big[V_{t+1}(S_{t+1}) | S_t = s, a_t \Big] \right) \tag{1}$$

  Recall, that S_{t+1} also depends on $a_t = a_t(s)$ (which we suppress in the notation).
- The optimal policy can be computed backward by **reward iteration**. Let a_t^* be a maximizing policy, that is a_t^* achieves C_t^* in Equation (1).
- One can now show that the **Bellman equation** holds, i.e.

$$V_t(s) = C_t^*(s) \quad t = 0, \dots, T.$$

- Under an additional (mild) structural assumption, one may verify that there always exist optimal policies π^* which can be obtained by maximizing the value function in each period (Theorem 2.3.8. in Bäuerle Rieder).

Algorithm

Step 0 Initialize by the terminal condition $V_T(S_T)$ and set $t = T - 1$

Step 1 Compute

$$V_t(s) = \sup_{a_t \in \mathscr{A}_t} \left(C_t(s, a_t) + \mathbb{E}\left[V_{t+1}(S_{t+1}) | S_t = s, a_t \right] \right)$$

for all $s \in \mathscr{S}$

Step 2 Decrement t and repeat Step 1 until $t = 0$

Infinite-time-horizon

- For this case several algorithms exist, to name **value iteration** and **policy iteration** which will not be discussed here, see Powell Section 3.3. ff.
- For more mathematical details (and there are many!) we refer to Powell, Bäuerle&Rieder and the excellent source Bertsekas&Shreve.

Approximate dynamic programming (ADP)

- While we introduce a nice theory beforehand, the core equation

$$\sup_{\pi} \mathbb{E}^{\pi}\left[\sum_{0=1}^{T} C_t(S_t, a_t)\right]$$

  my be intractable even for very small problems.
- ADP now offers a powerful set of strategies to solve these problems approximately.
- We have the problem of curse of dimensionality in **state space**, **outcome space** and **action space**.