Stochastic Filtering (SS2016) Exercise Sheet 9

Lecture and Exercises: JProf. Dr. Philipp Harms
Due date: July 1, 2016

9.1. Innovations approach for jumps observations

Let X solve the martingale problem associated to $A : \mathscr{D}(A) \subseteq B(\mathbb{R}^d) \to B(\mathbb{R}^d)$ and let Y be a Poisson process with rate $\lambda(X_-)$, i.e., $Y_t = N_{\int_0^t \lambda(X_{s-})ds}$, where N is a standard Poisson process independent of X and $\lambda : \mathbb{X} \to (0,\infty)$ is a measurable function. Assume that λ and λ^{-1} are bounded. Let I denote the innovations process $I_t = Y_t - \int_0^t \pi_t(\lambda)dt$.

a) Show that the $(\mathbb{F}(Y),\mathbb{P})$-compensator of $[I,I]_t$ is $\int_0^t \pi_{s-}(\lambda)ds$, i.e., the difference between the two is an $(\mathbb{F}(Y),\mathbb{P})$-local martingale.

b) Assume that $(I,\mathbb{F}(Y),\mathbb{P})$ has the strong property of predictable representation and show that the Kushner-Stratonovich equation holds, i.e.,

$$d\pi_t(f) = \pi_{t-}(Af)dt + \frac{\pi_{t-}(\lambda f) - \pi_{t-}(\lambda)\pi_{t-}(f)}{\pi_{t-}(\lambda)}dI_t, \qquad \forall f \in \mathscr{D}(A).$$

9.2. Correlated noise

The innovations approach allows one to treat filtering problems with correlated noise. Let X solve the martingale problem associated to $A : \mathscr{D}(A) \subseteq B(\mathbb{R}^d) \to B(\mathbb{R}^d)$, $M_t^f = f(X_t) - f(X_0) - \int_0^t Af(X_s)ds$, and $dY_t = h(X_t)dt + dW_t$, where W is d-dimensional Brownian motion. To model the dependence between X and W, assume that there are operators $C_i : \mathscr{D}(C_i) \subseteq B(\mathbb{X}) \to B(\mathbb{X})$, $i \in \{1,\dots,d\}$ such that

$$\langle M^f, W^i\rangle_t = \int_0^t (C_i f)(X_s)ds, \qquad \forall i \in \{1,\dots,d\}, \forall f \in \mathscr{D}(A) \cap \mathscr{D}(C_1) \cap \dots \cap \mathscr{D}(C_d).$$

The vector $(C_1 f,\dots,C_d f)$ is denoted by Cf.

a) Assume that $(I, \mathbb{F}(Y), \mathbb{P})$ has the strong property of predictable representation and show that the Kushner-Stratonovich equation takes the form

$$d\pi_t f = \pi_t(Af)dt + \big(\pi_t(fh) - \pi_t(f)\pi_t(h) + \pi_t(Cf)\big)\big(dY_t - \pi_t(h)dt\big),$$

for all $f \in \mathscr{D}(A) \cap \mathscr{D}(C_1) \cap \cdots \cap \mathscr{D}(C_d)$.

Hint: Adapt the proof of the Kushner-Stratonovich equation with uncorrelated noise.

b) What are the operators C in the case where X is a diffusion, say, $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$?

9.3. Asymptotics of the Kalman-Bucy filter

Let X and Y be real-valued processes solving the SDE

$$dX_t = \big(a_0 + a_1 X_t + a_2 Y_t\big)dt + \sum_{i=1}^{2} b_i \, dW_t^i,$$
$$dY_t = \big(\Lambda_0 + \Lambda_1 X_t + \Lambda_2 Y_t\big)dt + \sum_{i=1}^{2} B_i \, dW_t^i,$$

with normally distributed initial condition (X_0, Y_0), where W^1 and W^2 are two independent Wiener processes on $\mathbb{R}$ and where $B_1^2 + B_2^2 > 0$.

a) Explain on a conceptual level how the Kalman-Bucy filter with correlated noise can be derived, i.e.,

$$d\hat{X}_t = \big(a_0 + a_1 \hat{X}_t + a_2 Y_t\big)dt + \frac{b_1 B_1 + b_2 B_2 + \hat{\Sigma}_t \Lambda_1}{B_1^2 + B_2^2}\big(dY_t - (\Lambda_0 + \Lambda_1 \hat{X}_t + \Lambda_2 Y_t)dt\big),$$
$$\frac{d\hat{\Sigma}_t}{dt} = 2a_1 \hat{\Sigma}_t + b_1^2 + b_2^2 - \frac{(b_1 B_1 + b_2 B_2 + \hat{\Sigma}_t \Lambda_1)^2}{B_1^2 + B_2^2},$$

with initial conditions

$$\hat{X}_0 = \mathbb{E}[X_0 | Y_0], \qquad \hat{\Sigma}_0 = \mathrm{Var}(X_0 | Y_0).$$

b) Calculate $\lim_{t\to\infty}\hat{\Sigma}_t$ in the following special cases and compare the results:

(i) $dX_t = 0, dY_t = X_t dt + dW_t^2$

(ii) $dX_t = dW_t^1, dY_t = X_t dt + dW_t^2$

(iii) $dX_t = dW_t^2, dY_t = X_t dt + dW_t^2$

(iv) $dX_t = -dW_t^2, dY_t = X_t dt + dW_t^2$

9.4. Kalman filter for a model of population growth

We consider the following model for population growth with noisy observations:

$$dX_t = rX_t dt, \qquad dY_t = X_t dt + m dW_t,$$

with $X_0 \sim \mathcal{N}(b, a^2)$ and $Y_0 = 0$ for some constants $r, m, b, a > 0$.

a) Calculate $\lim_{t\to\infty}\hat{\Sigma}_t$. How is the asymptotic precision of the filter affected by the growth rate r?

b) Implement the Kálmán-Bucy filter for the model. In order to test your implementation approximate a path of $(X_t, Y_t)_{t\in[0,1]}$ using the Euler-Maruyama scheme. Use your implementation of the Kálmán-Bucy filter in order to recover the signal from the observation. Use the following parameters: $r = 0.5$, $m = 1$, $b = 1$, $a = 0.5$. What do you observe?