Stochastic Filtering (SS2016) Exercise Sheet 7

Lecture and Exercises: JProf. Dr. Philipp Harms
Due date: June 15, 2016

7.1. Kushner-Stratonovich and Zakai equation

Let X solve the martingale problem associated to $A : \mathscr{D}(A) \subseteq B(\mathbb{R}^d) \to B(\mathbb{R}^d)$ and let $dY_t = h(X_t)dt + dW_t$. It can be shown under certain conditions that the normalized filter satisfies the Kushner-Stratonovich equation

$$d\pi_t(f) = \pi_t(Af)dt + \big(\pi_t(fh) - \pi_t(f)\pi_t(h)\big)\big(dY_t - \pi_t(h)dt\big), \qquad f \in \mathscr{D}(A),$$

and the unnormalized filter satisfies the Zakai equation

$$d\rho_t(f) = \rho_t(Af)dt + \rho_t(fh)dY_t, \qquad f \in \mathscr{D}(A).$$

Deduce the Kushner-Stratonovich equation from the Zakai equation and the Kallianpur-Striebel formula $\pi_t(f) = \rho_t(f)/\rho_t(1)$.

Hint: Apply Itō's formula to $\rho_t(f)/\rho_t(1)$. What is the generator A applied to the constant function 1?

7.2. Kalman-Bucy filter

Let X and Y be real-valued processes solving the SDE

$$\begin{aligned} dX_t &= \big(a_0 + a_1 X_t\big)dt + b\,dW_t^1, \\ dY_t &= \big(\Lambda_0 + \Lambda_1 X_t\big)dt + B\,dW_t^2, \end{aligned}$$

with normally distributed initial condition (X_0, Y_0), where W^1 and W^2 are two independent Wiener processes on $\mathbb{R}$ and where $B^2 > 0$.

Assume that $\pi_t \sim \mathcal{N}(\hat{X}_t, \hat{\Sigma}_t)$ for some $\hat{X}_t$ and $\hat{\Sigma}_t$ and that the Kushner-Stratonovich equation holds for the functions $f(x) = x$ and $f(x) = x^2$. Show that

$$d\hat{X}_t = \big(a_0 + a_1\hat{X}_t\big)dt + \frac{\hat{\Sigma}_t\Lambda_1}{B^2}\big(dY_t - (\Lambda_0 + \Lambda_1\hat{X}_t)dt\big),$$
$$\frac{d\hat{\Sigma}_t}{dt} = 2a_1\hat{\Sigma}_t + b^2 - \frac{(\hat{\Sigma}_t\Lambda_1)^2}{B^2},$$

with initial conditions

$$\hat{X}_0 = \mathbb{E}[X_0|Y_0], \qquad \hat{\Sigma}_0 = \mathrm{Var}(X_0|Y_0).$$

Remark: The assumption that π_t is normally distributed and that the Kushner-Stratonovich equation holds for the functions $f(x) = x$ and $f(x) = x^2$ is justified in [1, Proposition 6.14].

7.3. Filtering the drift in a conditionally Gaussian model

We work on a finite time interval $[0,T]$. Let $(W_t)_{t\in[0,T]}$ be standard Brownian motion, X an independent random variable with finite exponential moments, and $Y_t = tX + \sigma W_t$, $t \in [0,T]$.

a) Define an equivalent probability measure $\tilde{\mathbb{P}}$ on $\mathscr{F}_T$ such that Y becomes a martingale and independent of X. Show that the unnormalized filter is given by

$$\rho_t(A) = \int_A \exp\left(\sigma^{-2}xY_t - \frac{1}{2}\sigma^{-2}x^2 t\right)\mu(dx).$$

Note that this shows that Y_t is a sufficient statistic for X.

b) Suppose in addition that X is normally distributed. Then π_t is normally distributed by a). Calculate the mean $\hat{X}_t$ and covariance $\hat{\Sigma}_t$ of π_t and verify that you get the same result as in Exercise 7.2.

7.4. Filtering the drift in the Black-Scholes model

Use the process Y_t of Exercise 7.3.b) as a model for log prices, where t is measured in years.

a) Download daily S&P 500 quotes (or any other quotes you are interested in) for a period of one year, ending today.[1] Propose a reasonable estimator for σ. What value do you get for your dataset?

b) Use the equation for $\hat{\Sigma}_t$ and realistic parameter values to derive 95% confidence intervals for the posterior distribution of X given $\mathscr{F}_t(Y)$. How many years of data are necessary to get reasonable precision?

References

[1] A. Bain and D. Crisan. *Fundamentals of Stochastic Filtering*. New York: Springer, 2009.

[1] Go to `http://finance.yahoo.com/q?s=gspc`, select 'Historical Prices', set the appropriate date range, click on 'Download to Spreadsheet' at the bottom of the page, and load the closing prices into your favorite software (MATLAB, R, ...).