Stochastic Filtering (SS2016) Exercise Sheet 6

Lecture and Exercises: JProf. Dr. Philipp Harms
Due date: June 8, 2016

6.1. Convergence of an Euler scheme for stochastic exponentials

Consider the stochastic exponential Y given by

$$dY_t = Y_t(\mu dt + \sigma dW_t), \qquad Y_0 = 1,$$

where μ and σ are constants. The Euler approximation of this SDE is

$$\Delta\hat{Y}_{n+1} = \hat{Y}_n(\mu\Delta t + \sigma\Delta\hat{W}_n), \qquad \hat{Y}_0 = 1,$$

where Δt is the step size, $\Delta\hat{Y}_{n+1} = \hat{Y}_{n+1} - \hat{Y}_n$, and $\Delta\hat{W}_n = W_{(n+1)\Delta t} - W_{n\Delta t}$.

a) Show that the local weak error is of second order, i.e.,

$$\limsup_{\Delta t\to 0} \left| \frac{\mathbb{E}[f(Y_{\Delta t})] - \mathbb{E}[f(\hat{Y}_1)]}{\Delta t^2} \right| < \infty$$

holds for any bounded smooth function f with bounded derivatives.

Hint: Expand $f(y)$ into a Taylor series around the point $y = 1$ and use Exercise 5.4 to calculate the distribution of $Y_{\Delta t}$.

b) The general theory tells us [3, Theorem 14.5.1] that the weak local errors aggregate to a weak global error of first order, i.e.,

$$\limsup_{\Delta t\to 0} \left| \frac{\mathbb{E}[f(Y_T)] - \mathbb{E}[f(\hat{Y}_{\lfloor T/\Delta t \rfloor})]}{\Delta t} \right| < \infty$$

holds for any bounded smooth function f with bounded derivatives and any $T \geq 0$. Verify this by numerically implementing the Euler-Maruyama scheme and checking the convergence rate.

6.2. Predictable and optional projections

Read Section 4.1 in [2].

a) Let X be an integrable random variable, seen as a constant process, and let M_t be the càdlàg version of the martingale $\mathbb{E}[X|\mathscr{F}_t]$. Show that the optional projection of X is M and the predictable projection of X is M_-.

   Remark: This is the key argument for proving the existence of optional projections. Note that this argument requires the usual conditions of the filtration.

b) What is the predictable projection of a deterministic and a Poisson process?

c) Let $\mathbb{G}$ be a sub-filtration of $\mathbb{F}$ and let M be an $\mathbb{F}$-martingale. Show that the $\mathbb{G}$-optional projection of M is a $\mathbb{G}$-martingale.

6.3. Increasing processes and projections

Read Section 4.2 in [2].

a) Let H be a bounded measurable raw process, i.e., we do not assume H to be adapted. Prove that the optional and predictable projection of the process $\int_0^t H_s ds$ is the process $\int_0^t {}^oH_s ds$.

6.4. Dual predictable and dual optional projections

Read Section 4.3 in [2] and in particular Definition 4.28.

a) What is the dual predictable projection of a Poisson process?

b) Show that the dual predictable projection of the process $\int_0^t H_s ds$ from Exercise 6.3 is the process $\int_0^t {}^oH_s ds$.

Remark

The boundedness, integrability, and monotonicity assumptions in [2, Section 4] can be weakened substantially. Increasing processes can be generalized to finite variation processes, integrability can be generalized to σ-integrability, and everything can be localized. A detailed treatment is provided in [1, Chapter V].

Please contact me if you can't download any of the given references.

References

[1] Sheng-wu He, Jia-gang Wang, and Jia-an Yan. *Semimartingale theory and stochastic calculus*. Taylor & Francis, 1992.

[2] Ashkan Nikeghbali et al. "An essay on the general theory of stochastic processes". In: *Probability Surveys* 3 (2006), pp. 345–412.

[3] Platen and Klöden. *Numerical Solution of Stochastic Differential Equations*. Springer Verlag, Berlin, 1995.