# Stochastic Filtering (SS2016) Exercise Sheet 5

Lecture and Exercises: JProf. Dr. Philipp Harms
Due date: June 1, 2016

## 5.1. Change of measure

Let $\mathbb{P}$ be the law of a HMM $(X,Y)$ with state kernel $P(x,dx')$ and observation kernel $K(x,dy) = \lambda(x,y)\phi(dy)$, where $\lambda$ is a positive function and $\phi$ a probability measure. Furthermore, let $\tilde{\mathbb{P}}$ be the law of a HMM $(X,Y)$ with the same state kernel $P(x,dx')$ and observation kernel $K(x,dy) = \phi(dy)$. Finally, let $\mathscr{F}_k = \sigma(X_{0:k}, Y_{0:k})$, $\mathscr{F}_\infty = \bigvee_{k\geq 0} \mathscr{F}_k$, and $\mathbb{F} = (\mathscr{F}_k)_{k\in\mathbb{N}}$.

a) Find an example where $\mathbb{P}|_{\mathscr{F}_k} \ll \tilde{\mathbb{P}}|_{\mathscr{F}_k}$ holds for each $k \in \mathbb{N}$, but not for $k = \infty$.

   Hint: Use the law of large numbers to construct an $\mathscr{F}_\infty$-measurable random variable which assumes one value $\mathbb{P}$-a.s. and another value $\tilde{\mathbb{P}}$-a.s.

b) Take $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be as in a) and let $\Lambda_k$ be the density of $\mathbb{P}|_{\mathscr{F}_k}$ with respect to $\tilde{\mathbb{P}}|_{\mathscr{F}_k}$. Then $\Lambda$ is a $\tilde{\mathbb{P}}$-martingale. Is it uniformly integrable?

## 5.2. Strong property of predictable representation

Let $M = (M_k)_{k\in\mathbb{N}}$ be a martingale on a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathscr{F}_k)_{k\in\mathbb{N}}$. We write $\mathscr{F}_k(M) = \sigma(M_0, \ldots, M_k)$, $\mathbb{F}(M) = (\mathscr{F}_k(M))_{k\in\mathbb{N}}$, and $\Delta M_k = M_k - M_{k-1}$. A process $H$ is called $\mathbb{F}$-predictable if $H_0$ is $\mathscr{F}_0$-measurable and $H_k$ is $\mathscr{F}_{k-1}$-measurable for each $k \in \mathbb{N}_{>0}$.

We say that the strong property of predictable representation holds for $(M, \mathbb{F}, \mathbb{P})$ if every $(\mathbb{F}, \mathbb{P})$-martingale $L$ can be written as $L_k = L_0 + \sum_{i=1}^k H_i \Delta M_i$, $k \in \mathbb{N}$, for some predictable process $H$.

a) Show that the strong property of predictable representation holds for $(M, \mathbb{F}(M), \mathbb{P})$, where $M_k = \sum_{i=1}^{k} \Delta M_k$ and $\Delta M_k$ is an i.i.d. sequence of random variables with uniform distribution on $\{-1, 1\}$.

Hint: You can focus on a single time step and construct the representing process $H$ explicitly.

Remark: In financial lingo, the strong property of predictable representation means that the market is complete.

### 5.3. Strong property of predictable representation

a) Come up with an example of an $(\mathbb{F}, \mathbb{P})$-martingale $M$ such that $(M, \mathbb{F}, \mathbb{P})$ does not have the strong property of predictable representation.

### 5.4. Stochastic exponentials

Let $X$ be an Itō process, i.e., $X$ satisfies

$$dX_t = \mu_t dt + \sigma_t dW_t$$

for some predictable processes $\mu, \sigma$ and Brownian motion $W$.

a) Show using Itō's formula that the process

$$Y_t = \exp\left(X_t - X_0 - \frac{1}{2}\langle X, X\rangle_t\right)$$

is a solution of the SDE

$$dY_t = Y_t dX_t, \qquad Y_0 = 1.$$

b) Show that $Y$ is a local martingale if $\mu = 0$.

c) Show that $Y$ is a martingale if $\mu = 0$ and $\sigma$ is bounded.

Hint: You can find this in any of the books [2, 3, 5, 4, 1].

## References


[1] Jean Jacod and Albert Shiryaev. *Limit theorems for stochastic processes*. Vol. 288. Springer Science & Business Media, 2013.

[2] Ioannis Karatzas and Steven Shreve. *Brownian motion and stochastic calculus*. Vol. 113. Springer Science & Business Media, 2012.

[3] Bernt Oksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.

[4] Philip E Protter. *Stochastic Differential Equations*. Springer, 2005.

[5] L Chris G Rogers and David Williams. *Diffusions, Markov processes and martingales*. Vol. 1 and 2. Cambridge University Press, 2000.