Stochastic Filtering (SS2016) Exercise Sheet 1

Lecture and Exercises: JProf. Dr. Philipp Harms
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1.1. Measures

Let $(\mathbb{X}, \mathscr{X})$ be a measurable space.

a) Give the definitions of the set $\mathscr{M}(\mathbb{X})$ of σ-finite measures and the subset $\mathscr{P}(\mathbb{X})$ of probability measures.

b) The spaces $\mathscr{M}(\mathbb{X})$ and $\mathscr{P}(\mathbb{X})$ are endowed with the σ-algebra generated by the mappings $\pi_A : \mu \mapsto \mu(A)$, $A \in \mathscr{X}$. Give the definition of this σ-algebra and show that $\mathscr{P}(\mathbb{X})$ is a measurable subset of $\mathscr{M}(\mathbb{X})$.

c) Show that the mappings $\pi_f : \mu \mapsto \int f d\mu$, where f runs through the set of bounded measurable functions on $\mathbb{X}$, generate the same σ-algebra.

1.2. Kernels

Fix two measurable spaces $(\mathbb{X}, \mathscr{X})$ and $(\mathbb{Y}, \mathscr{Y})$.

a) Give the definitions of kernels and probability kernels from $\mathbb{X}$ to $\mathbb{Y}$.

b) Show that the mapping $\delta : \mathbb{X} \times \mathscr{Y} \to [0,1], (x,A) \mapsto \mathbb{1}_A(f(x))$ is a probability kernel from $\mathbb{X}$ to $\mathbb{Y}$ iff $f : \mathbb{X} \to \mathbb{Y}$ is measurable.

c) Show that $P : \mathbb{X} \times \mathscr{Y} \to [0,1]$ is a probability kernel from $\mathbb{X}$ to $\mathbb{Y}$ iff $P : \mathbb{X} \to \mathscr{P}(\mathbb{Y})$ is measurable.

1.3. Conditioning

Let X and Y be random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with values in measurable spaces $(\mathbb{X}, \mathscr{X})$ and $(\mathbb{Y}, \mathscr{Y})$. Moreover, let $\mathscr{G}$ be a σ-algebra contained in $\mathscr{F}$.

a) What are the definitions of the regular conditional probabilities $P_{X|\mathscr{G}}$ and $P_{X|Y}$?

b) Are $\mathbb{P}[X \in \cdot|Y]$ and $P_{X|Y}(Y, \cdot)$ random probability measures, i.e., probability kernels on $(\Omega, \mathscr{F})$? Which properties need to be verified?

c) Show that $P_{X|\sigma(Y)}(\omega, A) = P_{X|Y}(Y(\omega), A)$ holds identically.

1.4. Bayes' formula

Let X and Y be random variables with values in measurable spaces $(\mathbb{X}, \mathscr{X})$ and $(\mathbb{Y}, \mathscr{Y})$, respectively. Assume that the law of (X,Y) can be written in the form $\Lambda(x,y)\mu_X(dx)\mu_Y(dy)$ for some non-negative measurable function $\Lambda : \mathbb{X} \times \mathbb{Y} \to \mathbb{R}$ and some probability measures $\mu_X \in \mathscr{P}(\mathbb{X})$ and $\mu_Y \in \mathscr{P}(\mathbb{Y})$.

a) Show that the following is a regular conditional probability of X given Y:

$$P_{X|Y}(y, A) = \frac{\int_A \Lambda(x,y)\mu_X(dx)}{\int \Lambda(x,y)\mu_X(dx)}$$

(In the expression above, the fraction is set to zero if the denominator vanishes.)

b) Use a) to calculate the regular conditional probability of X given Y if (X,Y) is multivariate Gaussian.

Hint: the solution is $P_{X|Y}(y, dx) \sim \mathscr{N}(\hat{X}, \hat{\Sigma})$ with

$$\hat{X} = \mathbb{E}[X] + \operatorname{Cov}(X,Y)\operatorname{Var}(Y)^{-1}(y - \mathbb{E}[Y]),$$
$$\hat{\Sigma} = \operatorname{Var}(X) - \operatorname{Cov}(X,Y)\operatorname{Var}(Y)^{-1}\operatorname{Cov}(Y,X),$$

where $\operatorname{Var}(Y)^{-1}$ is the generalized inverse of the (co-)variance matrix of Y.

1.5. Hidden Markov Models

Let (X,Y) be a Markov process. Show that the following statements are equivalent:

a) The transition kernel of (X,Y) is of the form $P(x,dx')K(x',dy')$.

b) X is Markov in the filtration generated by (X,Y), and Y_{k+1} is independent of (X_k,Y_k) conditional on X_{k+1}, for each $k \in \mathbb{N}$.

c) X_{k+1} is independent of Y_k conditional on X_k, and Y_{k+1} is independent of (X_k,Y_k) conditional on X_{k+1}, for each $k \in \mathbb{N}$.

1.6. Hidden Markov Models

a) Show that the process (X,Y) defined by

$$X_{k+1} = f(X_k, \alpha_k), \qquad Y_{k+1} = g(X_{k+1}, \beta_k), \quad k = 0,1,2,\ldots$$

is a Hidden Markov Model, where $f : \mathbb{X} \times [0,1] \to \mathbb{X}$ and $g : \mathbb{X} \times [0,1] \to \mathbb{Y}$ are measurable functions, and where $(\alpha_k)_{k\in\mathbb{N}}$ and $(\beta_k)_{k\in\mathbb{N}}$ are independent sequences of i.i.d. $[0,1]$-valued random variables.

b) Conversely, show that any Hidden Markov Model is of this form if $\mathbb{X}$ and $\mathbb{Y}$ are Borel spaces.

Hint. You may use [1, Theorem 6.10] and the discussions preceding this theorem.

References

[1] Olav Kallenberg. *Foundations of modern probability*. 2nd ed. Springer Verlag, New York, 2002.