Analysis of risk bounds in partially specified additive factor models

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Abstract

The study of worst case scenarios for risk measures (e.g. the Value at Risk) when the underlying risk vector (or portfolio of risks) is not completely specified is a central topic in the literature on robust risk measurement. In this paper we discuss partially specified factor models as introduced in Bernard, Rüschendorf, Vanduffel and Wang (2017) in more detail for the class of additive factor models which admit more explicit results. These results allow to describe in more detail the reduction of risk bounds obtainable by this method in dependence on the degree of positive resp. negative dependence induced by the systematic risk factors. The insight may help in applications of this reduction method to get a better qualitative impression on the range of influence of the partially specified factor structure.

Key-words: risk factor models, risk bounds, dependence uncertainty, Value at Risk

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1 Introduction

There is a rich literature on finding bounds for the Value at Risk (VaR) or other risk measures of a portfolio under the assumption that all marginal distributions are known, but the dependence of the portfolio is either unknown or only partially known. Moment bounds for VaR (which are intimately connected with distributional bounds) or for the Tail Value at Risk (TVaR) based on some moment informations on the distribution have been studied intensively in the insurance literature by various authors as Kaas and Goovaerts (1986), Denuit et al. (1999), de Schepper and Heijnen (2010), Hürlimann (2002, 2008), Goovaerts et al. (2011), Bernard, Rüschendorf and Vanduffel (2017), Bernard et al. (2018), Tian (2008), Cornilly et al. (2018). Specifically Hürlimann (2002) derived analytical bounds for VaR and TVaR under knowledge of the mean, variance, skewness and kurtosis. Risk

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bounds with pure marginal information were found to be often too wide in order to be useful for applications (see Embrechts and Puccetti (2006a) and Embrechts et al. (2013, 2014)). Related aggregation-robustness and model uncertainty for risk measures are also investigated in Embrechts et al. (2015). Several approaches to add to marginal information some dependence information have been discussed in ample literature (see Puccetti and Rüschendorf (2012a,b, 2013), Bernard and Vanduffel (2015), Bernard, Rüschendorf and Vanduffel (2017), Bernard, Rüschendorf, Vanduffel and Wang (2017), Bignozzi et al. (2015), Rüschendorf and Witting (2017), Puccetti et al. (2017)). For some surveys on these developments see Rüschendorf (2017a,b).

Partially specified factor models are introduced in Bernard, Rüschendorf, Vanduffel and Wang (2017). The aim of that paper is to introduce for a risk model additionally to the marginals of a risk vector structural information given by a systematic risk factor, which allows to reduce the wide dependence uncertainty (DU) as in pure marginal models.

Partially specified risk factor models (PSFM) are of the form

$$X_i = f_i(Z, \varepsilon_i), \quad 1 \leqslant i \leqslant n, \tag{1.1}$$

where Z is a real systematic risk factor and (ε_i) are idiosyncratic risk factors. The distributions of (Z, ε_i) are assumed to be known but in contrast to the usual factor models the joint distribution of (ε_i) given Z is not specified; in particular conditional given Z the (ε_i) are not assumed to be independent. By construction the marginals F_i of X_i are determined. In the paper of Bernard, Rüschendorf, Vanduffel and Wang (2017) it is shown that PSFM are a particularly flexible and effective class of models to reduce risk bounds compared to those in the marginal model $\mathcal{F}(F_1, \dots, F_n)$, using only marginal information. Further, that paper develops techniques how to compute the risk bounds numerically. Several examples in the paper give an impression of the kind and of the magnitude of reduction of risk bounds resulting from the additional structural information.

The present paper considers a special but informative class of PSFM, the class of partially specified additive factor models (PSAFM), which allows to determine analytically the risk bounds of interest and to describe the magnitude of reduction compared to the marginal models in explicit form. As a result it becomes evident that in PSFM, where the risk factor Z induces positive dependence essentially the lower risk bounds are improved compared to the marginal model, while in the case that Z induces negative dependence, the upper risk bounds improve essentially, while there is only a minor effect to the lower risk bounds. All intermediate types of reduction are possible within this class of models. As a result the considered additive models allow to get a better understanding of this behaviour of PSFM which was observed and described in Bernard, Rüschendorf, Vanduffel and Wang (2017) in some specific examples.

2 Partially specified additive factor models

Partially specified risk factor models (PSFM) as defined in (1.1) are determined in functional form by functions of the systematic risk factor Z and the idiosyncratic risk factors ε_i. Motivated by the simple form of standard additive factor models we consider in this paper as model class partially specified additive factor models (PSAFM) (in case $n = 2$)

given by

$$X_1 = Z_1 \oplus \varepsilon_1, \quad X_2 = Z_2 \oplus \varepsilon_2. \tag{2.1}$$

Here Z_i are systematic risk factors. ε_i are idiosyncratic risk factors independent of (Z_1, Z_2) and $\oplus$ denotes an independent sum. For the (PSAFM) in (2.1) it is generally assumed that the joint distribution of (Z_1, Z_2) and also the marginal distributions of ε_1, ε_2 are know. It is further assumed that ε_i is independent of (Z_1, Z_2). On the other hand the joint distribution of $(\varepsilon_1, \varepsilon_2)$ is unknown and it is not assumed that $(\varepsilon_1, \varepsilon_2)$ is independent of (Z_1, Z_2). We comment in Section 5 on extensions to the case $n \geqslant 2$.

(2.1) includes the PSFM in (1.1) in the case where $X_i = f_i(Z) + \varepsilon_i$ by choosing $Z_i = f_i(Z)$. This model assumption in (2.1) allows to consider two general dependent systematic risk factors Z_1, Z_2 and also allows an easy extension to multiplicative models of the form $X_i = Z_i \varepsilon_i$.

Let $\varepsilon_i \sim G_i$, $Z_i \sim H_i$, then as consequence of the specification of the (PSAFM) the marginal distribution F_i is given by the convolution of H_i and G_i, i.e.

$$F_i = H_i * G_i.$$

In the particular case when $Z_1 = Z_2 = Z \sim G$ this amounts to the partially specified additive factor model

$$X_1 = Z \oplus \varepsilon_1, \quad X_2 = Z \oplus \varepsilon_2 \tag{2.2}$$

with a systematic risk factor Z inducing positive dependence.

In order to demonstrate the effects of the risk factors on the additive model we consider in the following in typical examples the case of equal distribution where $Z_1 \sim Z_2$ and $\varepsilon_1 \sim \varepsilon_2$. When determining risk bounds for tails, we consider the case of asymptotically equivalent tails, i.e. $P(Z_1 > t) \sim P(Z_2 > t)$, $P(\varepsilon_1 > t) \sim P(\varepsilon_2 > t)$. In particular we investigate the risk bounds for $X_1 + X_2$ in the case when the systematic risk part is dominating and the case when both risk parts are of similar order.

Define the tail risk bounds in the partially specified additive factor model (PSAFM) (2.1) by

$$\begin{aligned} M^f(s) &= \sup\{P(X_1 + X_2 \geqslant s); \quad (X_1, X_2) \text{ satisfy (2.1)}\} \\ m^f(s) &= \inf\{P(X_1 + X_2 \geqslant s); \quad (X_1, X_2) \text{ satisfy (2.1)}\}. \end{aligned} \tag{2.3}$$

Let

$$\begin{aligned} M(s) &= \sup\{P(X_1 + X_2 \geqslant s); \quad X_i \sim F_i, i = 1, 2\} \\ m(s) &= \inf\{P(X_1 + X_2 \geqslant s); \quad X_i \sim F_i, i = 1, 2\} \end{aligned} \tag{2.4}$$

be the corresponding marginal tail risk bounds. The associated dependence uncertainty spreads are given by

$$DU^f(s) = M^f(s) - m^f(s) \quad \text{resp.} \quad DU(s) = M(s) - m(s). \tag{2.5}$$

Besides the tail risk we also consider risk bounds for law invariant convex risk measures ϱ like TVaR (expected shortfall) and the related convex ordering results.

For a random variable X with distribution function F define the Value at Risk at level α by the right continuous inverse of F, i.e.

$$\mathrm{VaR}_\alpha(X) = \inf\{\gamma; P(X \leqslant \gamma) \geqslant \alpha\} = F^{-1}(\alpha).$$

Then $\mathrm{VaR}_\alpha(X) = \inf\{\gamma; P(X \geqslant \gamma) < 1-\alpha\} = \overline{F}^{-1}(1-\alpha)$ is identical to the right continuous inverse of the survival function $\overline{F}(\gamma) = P(X \geqslant \gamma)$ at level $1-\alpha$.

For the marginal (unconstrained) case the tail-risk bounds and the corresponding Value at Risk (VaR) bounds are given in the following proposition (see Makarov (1981) and Rüschendorf (1982)). Let for marginal distribution functions F_1, F_2,

$$\overline{\mathrm{VaR}}_\alpha = \sup\{\mathrm{VaR}_\alpha(X_1+X_2); X_i \sim F_i\} \quad \text{and} \quad \underline{\mathrm{VaR}}_\alpha = \inf\{\mathrm{VaR}_\alpha(X_1+X_2); X_i \sim F_i\}$$

denote the VaR bounds in the marginal model.

Proposition 2.1. *For fixed marginals F_1, F_2 it holds that:*

a)
$$M(s) = \inf_x(\overline{F}_1(x+) + \overline{F}_2(s-x)) \tag{2.6}$$
$$m(s) = \sup_x(\overline{F}_1(x+) + \overline{F}_2(s-x)) - 1, \tag{2.7}$$

where $\overline{F}_i(x) = P(X_i \geqslant x)$ and $\overline{F}_i(x+) = P(X_i > x)$.

b) For $\alpha \in (0,1)$ holds

$$\overline{\mathrm{VaR}}_\alpha = \inf_{\alpha \leqslant u \leqslant 1}(F_1^{-1}(u) + F_2^{-1}(1+\alpha-u)) \tag{2.8}$$
$$\text{and} \quad \underline{\mathrm{VaR}}_\alpha = \sup_{0 \leqslant u \leqslant \alpha}(F_1^{-1}(u) + F_2^{-1}(\alpha-u)).$$

Remark 2.2. *a) The upper tail risk bound $M(s)$ is attained for*

$$X_1 = F_1^{-1}(U), \quad X_2 = F_2^{-1}(U)1_{\{U \leqslant \alpha\}} + F_2^{-1}(1+\alpha-U)1_{\{U \geqslant \alpha\}} \tag{2.9}$$

which are negatively dependent (antimonotonic) in the upper α-part, where $\alpha = 1 - M(s)$. The lower bound is attained for $X_1 = F_1^{-1}(U)$, $X_2 = F_2^{-1}(\alpha-U)1_{\{U \leqslant \alpha\}} + F_2^{-1}(1+\alpha-U)1_{\{U \geqslant \alpha\}}$ (see Rüschendorf (1982)).

b) The upper VaR *bound $\overline{\mathrm{VaR}}_\alpha$ can be represented as the right continuous inverse of the tail risk bound M at level $1-\alpha$, i.e.*

$$\overline{\mathrm{VaR}}_\alpha = \inf\{\gamma; M(\gamma) < 1-\alpha\} = M^{-1}(1-\alpha).$$

c) If F_i have densities g_i, decreasing on $[x_0, \infty)$ then for $s \geqslant s_0$ the infimum in (2.6) *is attained at the unique solution u^* of*

$$g_1(u^*) = g_2(s-u^*), \tag{2.10}$$

as results from a first order condition (see Embrechts and Puccetti (2006b) and Puccetti et al. (2016)). This gives an easy recipe to calculate the tail risk bound in (2.6).

For real random variables X, Y the convex order $X \leqslant_{\mathrm{cx}} Y$ is defined by $Ef(X) \leqslant Ef(Y)$ for all convex real functions F such that $f(X)$ and $f(Y)$ are integrable. The ordering results for convex, law invariant risk measures ϱ are consequences of the following well-known convex ordering result in the unconstrained case (see Meilijson and Nadas (1979)).

Proposition 2.3 (Convex ordering in marginal models)**.** *For $X_i \sim F_i$, $i = 1, 2$ holds*

$$F_1^{-1}(U) + F_2^{-1}(1-U) \leqslant_{cx} X_1 + X_2 \leqslant_{cx} F_1^{-1}(U) + F_2^{-1}(U) \tag{2.11}$$

where $U \sim U(0,1)$.

The upper bound in (2.11) is given by the comonotonic pair

$$(X_1, X_2^c) = (F_1^{-1}(U), F_2^{-1}(U)). \tag{2.12}$$

The lower bound is given by the antimonotonic (countermonotonic) pair

$$(X_1, X_2^{cm}) = (F_1^{-1}(U), F_2^{-1}(1-U)). \tag{2.13}$$

Generally for Z_1, Z_2 the comonotonic resp. countermonotonic version of Z_2 w.r.t. Z_1 are denoted by Z_2^c Z_2^{cm}.

As consequence of Proposition 2.3 one obtains in the PSAFM (2.1):

Theorem 2.4 (Convex ordering of unconstrained and constrained PSAFM)**.**
a) For a PSAFM model $X_i = Z_i \oplus \varepsilon_i$, $i = 1, 2$ holds:

$$(Z_1 + Z_2) \oplus (\varepsilon_1 + \varepsilon_2^{cm}) \leqslant_{cx} X_1 + X_2 \leqslant_{cx} (Z_1 + Z_2) \oplus (\varepsilon_1 + \varepsilon_2^c) \tag{2.14}$$

b) For the unconstrained factor model $X_i = Z_i \oplus \varepsilon_i$, $i = 1, 2$ holds

$$(Z_1 \oplus \varepsilon_1) + (Z_2 + \varepsilon_2)^{cm} \leqslant_{cx} X_1 + X_2 \leqslant_{cx} (Z_1 \oplus \varepsilon_1) + (Z_2 \oplus \varepsilon_2)^c. \tag{2.15}$$

c) The upper and lower bounds in a) and b) are sharp.

Proof. a) By assumption ε_i is independent of $Z = (Z_1, Z_2)$ and thus $\varepsilon_i \mid Z = z \sim G_i$, $i = 1, 2$ for all z. Thus the conditional distribution of $(\varepsilon_1, \varepsilon_2) \mid Z = z$ is in the Fréchet class $\mathcal{F}(G_1, G_2)$ for all z. This implies by Proposition 2.3

$$\varepsilon_1 + \varepsilon_2^{cm} \leqslant_{\mathrm{cx}} (\varepsilon_1 + \varepsilon_2 \mid Z = z) \leqslant_{\mathrm{cx}} \varepsilon_1 + \varepsilon_2^c \tag{2.16}$$

and using that $(Z_1 + Z_2 + \varepsilon_1 + \varepsilon_2 \mid Z = z) \stackrel{d}{=} (z_1 + z_2 + \varepsilon_1 + \varepsilon_2 \mid Z = z)$ this yields

$$z_1 + z_2 + \varepsilon_1 + \varepsilon_2^{cm} \leqslant_{\mathrm{cx}} (z_1 + z_2 + \varepsilon_1 + \varepsilon_2 \mid Z = z) \leqslant_{\mathrm{cx}} z_1 + z_2 + \varepsilon_1 + \varepsilon_2^c. \tag{2.17}$$

Since convex ordering is stable under mixing this implies that

$$(Z_1 + Z_2) \oplus (\varepsilon_1 + \varepsilon_2^{cm}) \leqslant_{\mathrm{cx}} Z_1 + Z_2 + \varepsilon_1 + \varepsilon_2 \leqslant_{\mathrm{cx}} (Z_1 + Z_2) \oplus (\varepsilon_1 + \varepsilon_2^c). \tag{2.18}$$

b) and c) follow directly from Proposition 2.3. $\square$

Remark 2.5. *a) For a general PSFM as in* (1.1) *it has been shown in Bernard, Rüschendorf, Vanduffel and Wang (2017) that the worst case distribution is given by the conditional comonotonic random vector (given Z). This coincides in the partially specified additive factor model with the upper bound in* (2.14)*.*

The simple argument also extends to the lower bound in (2.14)*. It is remarkable that w.r.t. convex ordering the best and the worst cases are attained in the case where the sum of the idiosyncratic risks $\varepsilon_1 + \varepsilon_2$ is independent of the systematic risk factor (Z_1, Z_2) while by assumption only independence of ε_i, (Z_1, Z_2) for $i = 1, 2$ holds.*

b) The distribution function of the worst case pair w.r.t. convex ordering in the unconstrained case in (2.15) *is given by*

$$\widehat{F}^{uc}(x_1, x_2) = \min\{H_1 * G_1(x_1), H_2 * G_2(x_2)\}; \tag{2.19}$$

that of the best case (lower bound) by

$$\check{F}^{uc}(x_1, x_2) = (H_1 * G_1(x_1) + H_2 * G_2(x_2) - 1)_+.$$

In comparison for the constrained case in (2.14) *holds*

$$\widehat{F}^{cs}(x_1, x_2) = \int \min\{G_1(x_1 - z), G_2(x_2 - z)\} dG(z) \tag{2.20}$$

and

$$\check{F}^{cs}(x_1, x_2) = \int (G_1(x_1 - z) + G_2(x_2 - z) - 1)_+ dG(z), \quad \text{where } G \sim Z_1 + Z_2.$$

It follows that

$$\check{F}^{uc}(x_1, x_2) \geqslant \check{F}^{cs}(x_1, x_2), \qquad \widehat{F}^{cs}(x_1, x_2) \leqslant \widehat{F}^{uc}(x_1, x_2). \tag{2.21}$$

c) The argument for the proof of Theorem 2.4 reveals an even stronger ordering result. In the PSAFM holds:

$$(Z_1 \oplus \varepsilon_1, Z_2 \oplus \varepsilon_2^{cm}) \leqslant_{sm} (X_1, X_2) \leqslant_{sm} (Z_1 \oplus \varepsilon_1, Z_2 \oplus \varepsilon_2^{c}), \tag{2.22}$$

where $\leqslant_{sm}$ denotes the supermodular ordering.

If $Z_2 = Z_2^c$ is comonotonic to Z_1 then the systematic risk factors induce positive dependence and from (2.22) *it follows that in particular the lower risk bound is improved since*

$$(Z_1 \oplus \varepsilon_1, Z_2 \oplus \varepsilon_2^{cm}) \leqslant_{sm} (Z_1 \oplus \varepsilon_1, Z_2^c \oplus \varepsilon_2^{cm}).$$

Similarly, if $Z_2 = Z_2^{cm}$, then the upper risk bound is improved.

In the unconstrained case holds:

$$(X_1, X_2^{cm}) \leqslant_{sm} (X_1, X_2) \leqslant_{sm} (X_1, X_2^c). \tag{2.23}$$

d) It is obvious that the upper and lower bounds (2.14) *in the PSFM case improve upon the bounds* (2.15) *in the unconstrained case in convex ordering.*

Next we determine the worst and best case tail risk bounds under the structural assumption given by the PSAFM model. Let $(\varepsilon_1^z, \varepsilon_2^{z,c})$ and $(\varepsilon_1^z, \varepsilon_2^{z,cm})$ be a worst case pair respectively a best case pair in the Fréchet class $\mathcal{F}(G_1, G_2)$ satisfying

$$P(\varepsilon_1 + \varepsilon_2 \geqslant s - (z_1 + z_2)) = \max \quad \text{respectively} \quad = \min, \quad z = (z_1, z_2).$$

By Proposition 2.1 holds

$$\begin{aligned} M_\varepsilon(s - (z_1 + z_2)) &= \sup\{P(\varepsilon_1 + \varepsilon_2 \geqslant s - (z_1 + z_2)); \varepsilon_i \sim G_i\} \\ &= P(\varepsilon_1^z + \varepsilon_2^{z,cm} \geqslant s - (z_1 + z_2)) \\ &= \inf_x \{\overline{G}_1(x+) + \overline{G}_2(s - (z_1 + z_2) - x)\} - 1. \end{aligned} \tag{2.24}$$

Let $Z = (Z_1, Z_2)$ be independent of ε_1^z, $\varepsilon_2^{z,c}$ and define

$$\varepsilon_1^* := \varepsilon_1^Z; \;\; \varepsilon_2^{*,c} := \varepsilon_2^{Z,c}, \;\; \varepsilon_i^{*,cm} = \varepsilon_i^{Z,cm}; \quad X_i^{*,c} = Z_i + \varepsilon_i^{*,c}; \;\; X_i^{*,cm} = Z_i + \varepsilon_i^{*,cm}, \quad i = 1, 2. \tag{2.25}$$

Theorem 2.6 (Tail risk bounds in PSAFM)**.** *For a partially specified additive factor model $X_i = Z_i \oplus \varepsilon_i$, $i = 1, 2$ holds:*

$$\begin{aligned} \underline{F}^{cs}(s) := P(Z_1 + Z_2 + \varepsilon_1^* + \varepsilon_2^{*,cm} \geqslant s) &\leqslant P(X_1 + X_2 \geqslant s) \\ &\leqslant P(Z_1 + Z_2 + \varepsilon_1^* + \varepsilon_2^{*,c} \geqslant s) := \overline{F}^{cs}(s). \end{aligned} \tag{2.26}$$

Proof. As shown in the proof of Theorem 2.4 the conditional distributions of $(\varepsilon_1, \varepsilon_2) \mid Z = z$, $Z = (Z_1, Z_2)$ belong to the Fréchet class $\mathcal{F}(G_1, G_2)$. As consequence this implies by conditioning

$$\begin{aligned} P(X_1 + X_2 \geqslant s) &= P(Z_1 + Z_2 + \varepsilon_1 + \varepsilon_2 \geqslant s) \\ &= \int P(\varepsilon_1 + \varepsilon_2 \geqslant s - z \mid (Z_1, Z_2) = z) dG(z) \\ &\leqslant \int P(\varepsilon_1^{z,c} + \varepsilon_2^{z,c} \geqslant s - z \mid (Z_1, Z_2) = z) dG(z) \\ &= P(Z_1 + Z_2 + \varepsilon_1^{*,c} + \varepsilon_2^{*,c} \geqslant s) = \overline{F}^{cs}(s), \;\; G \sim (Z_1, Z_2). \end{aligned} \tag{2.27}$$

For the inequality in (2.27) the Fréchet bounds for the conditional distributions are used.

Similarly we get for the lower bounds

$$\begin{aligned} P(X_1 + X_2 \geqslant s) &= \int P(\varepsilon_1 + \varepsilon_2 \geqslant s - (z_1 + z_2) \mid Z = z) dG(z), \quad G \sim (Z_1, Z_2) \\ &\geqslant \int P(\varepsilon_1^{z,cm} + \varepsilon_2^{z,cm} \geqslant s - (z_1 + z_2) \mid Z = z) dG(z) \\ &= P(Z_1 + Z_2 + \varepsilon_1^{*,cm} + \varepsilon_2^{*,cm} \geqslant s) =: \underline{F}^{cs}(s). \end{aligned} \tag{2.28}$$

$\square$

Note that $\varepsilon_i^{*,c}$ and $\varepsilon_i^{*,cm}$ depend on s and, therefore, also the best case pair

$$(X_1, X_2) = (Z_1 + \varepsilon_1^*, Z_2 + \varepsilon_2^{*,cm}) \tag{2.29}$$

and the worst case pair

$$(X_1, X_2) = (Z_1 + \varepsilon_1^*, Z_2 + \varepsilon_2^*) \tag{2.30}$$

depend on s.

For concrete classes of distributions the best and worst case tail risks can be calculated explicitly.

Example 2.7 (PSAFM with Pareto tails)**.** *Let $\varepsilon_1 \sim \varepsilon_2$ be Pareto(2) with tail risk $\overline{G}_1(s) = \overline{G}_2(s) = P(\varepsilon_1 \geqslant s) = \frac{1}{s^2}$, $s \geqslant 1$. Then with $z = z_1 + z_2$*

$$A(s,z) := P(\varepsilon_1^z + \varepsilon_2^{z,c} \geqslant s - z) = \inf_x \{\overline{G}_1(x+) + \overline{G}_2(s - z - x)\}$$

and similarly for the lower bound $B(s,z)$. With $\overline{G}_1(x) = \begin{cases} \frac{1}{x^2}, & x \geqslant 1 \\ 1, & x < 1 \end{cases}$ holds

$$\begin{aligned} A(s) := \inf_x (\overline{G}_1(x+) + \overline{G}_2(s-x)) &= \begin{cases} \frac{8}{s^2} & \text{if } 1 \leqslant x \leqslant s-1 \\ 1 + \frac{1}{(s-1)^2} & \text{if } x > s-1 \text{ or } x < 1 \end{cases} \\ &= \min\left\{\frac{8}{s^2}, 1\right\}, \end{aligned}$$

taking the minima over $1 \leqslant x \leqslant s-1$ etc. Similarly,

$$B(s) = \sup_x (\overline{G}_1(x+) + \overline{G}_2(s-x)) - 1 = \min\left\{\frac{2}{(s-1)^2}, 1\right\}.$$

For $s \geqslant \sqrt{8}$ holds $A(s) = \frac{8}{s^2}$ and for $s \geqslant \sqrt{2} + 1$ holds $B(s) = \frac{2}{(s-1)^2}$.

This implies that

$$A(s,z) = \begin{cases} \frac{8}{(s-z)^2} & \text{for } z \leqslant \frac{1}{2}(s - \sqrt{8}) \\ 1 & \text{else} \end{cases} \tag{2.31}$$

and

$$B(s,z) = \begin{cases} \frac{2}{(s-z-1)^2} & \text{for } z \leqslant \frac{1}{2}(s - \sqrt{2} - 1) \\ 1 & \text{else.} \end{cases}$$

As a consequence this implies with $G \sim Z_1 + Z_2$

$$\overline{F}^{cs}(s) = \int_{-\infty}^{\frac{1}{2}(s-\sqrt{8})} \frac{8}{(s-2z)^2} dG(z) + \overline{G}\left(\frac{1}{2}(s - \sqrt{8})\right) \tag{2.32}$$

and

$$\underline{F}^{cs}(s) = \int_{-\infty}^{\frac{1}{2}(s-\sqrt{2}-1)} \frac{2}{(s-z-1)^2} dG(z) + \overline{G}\left(\frac{1}{2}(s - \sqrt{2} - 1)\right) \tag{2.33}$$

which can be calculated explicitly or numerically for concrete distributions G. □

As a result the formulas in (2.26) and (2.28) imply that

$$\underline{F}^{uc}(s) \leqslant \underline{F}^{cs}(s) \leqslant \overline{F}^{cs}(s) \leqslant \overline{F}^{uc}(s) \tag{2.34}$$

and thus also

$$\underline{\mathrm{VaR}}_{\alpha}^{uc} \leqslant \underline{\mathrm{VaR}}^{cs} \leqslant \overline{\mathrm{VaR}}^{cs} \leqslant \overline{\mathrm{VaR}}_{\alpha}^{uc}. \tag{2.35}$$

The formulas in (2.26) and in (2.32), (2.33) give a clear and almost explicit expression for the influence of the positive and of the negative dependence of the systematic risk factors Z_1, Z_2 on the worst case tail risk bounds as well as on the best case tail risk bounds. The results depend only on the distribution G of the sum $Z = Z_1 + Z_2$ of the systematic risks.

For convex risk measures the influence of positive and negative dependence of Z_1, Z_2 is more directly to describe by corresponding convex ordering results. This is the subject of the following Sections 3 and 4.

3 Comparison of PSAFM and marginal model in positive dependent factor case

Based on the determination of best and worst case distributions for the convex ordering in Theorem 2.4 we compare in this section the tail behaviour of the best and worst cases between the marginal model and the PSAFM model $X_i = Z + \varepsilon_i$, $i = 1, 2$, i.e. in the positive dependent factor case.

3.1 Dominant systematic risk

We consider first the case where the effect of the systematic risk factor is dominating the effect of the idiosyncratic risk.

Example 3.1 (Normal risks). *In this example the case where the systematic risks and the idiosyncratic risks are normal and the systematic risks are dominating is considered. As example let $Z_1 = Z_2 = Z \sim N(a, \sigma_1^2)$ and $\varepsilon_i \sim N(b, \sigma_2^2)$ where $a > b$. This implies for the constrained case the worst case pair*

$$\begin{aligned} W^{cs} &= 2Z \oplus (\varepsilon_1 + \varepsilon_2^c) = N(2a, 4\sigma_1^2) \oplus N(2b, 4\sigma_2^2) \\ &= N(2a + 2b, 4\sigma^2), \quad \sigma^2 = \sigma_1^2 + \sigma_2^2. \end{aligned} \tag{3.1}$$

The best case pair is given by

$$B^{cs} = 2Z \oplus (\varepsilon_1 + \varepsilon_2^{cm}) = 2Z + 2b = N(2a + 2b, 4\sigma_1^2). \tag{3.2}$$

For the unconstrained case the worst case sum is given by

$$W^{uc} = (Z \oplus \varepsilon_1) + (Z \oplus \varepsilon_2)^c \sim 2N(a + b, 4\sigma^2) \sim N(2(a + b), 4\sigma^2) \tag{3.3}$$

and the best case sum is constant,

$$B^{uc} = 2(a+b). \tag{3.4}$$

As a result the lower bounds improve strongly in convex order, while the upper bounds remain unchanged. Similarly one gets for the corresponding tail risks

$$\overline{F}_{B^{uc}}(s) \leqslant \overline{F}_{B^{cs}}(s) \leqslant \overline{F}_{W^{cs}}(s) \leqslant \overline{F}_{W^{uc}}(s). \tag{3.5}$$

□

Typical distributions in insurance applications are Pareto type distributions. The following proposition describes the worst and the best case risks in the case of positive dependent dominating systematic Paretian risk factor with $X_i = Z + \varepsilon_i$, $i = 1, 2$. As in (3.1), (3.2) worst and best case pairs are given by

$$W^{cs} = 2Z \oplus (\varepsilon_1 + \varepsilon_2^c), \quad B^{cs} = 2Z \oplus (\varepsilon_1 + \varepsilon_2^{cm}).$$

Proposition 3.2 (Paretian tails, dominating systematic risks)**.** *Assume that the risks have Paretian tails with dominating systematic risk, i.e.*

$$\overline{F}_Z(s) \sim s^{-\beta}, \quad \overline{F}_{\varepsilon_i}(s) \sim s^{-\gamma}, \quad s \geqslant 1, \quad 1 < \beta < \gamma.$$

Then it holds:

a) The tail risks of the worst resp. best case distributions (in convex order) are

$$\begin{aligned} \overline{F}_{W^{cs}}(s) &\sim 2^\beta s^{-\beta} + 2^\gamma s^{-\gamma} \sim 2^\beta s^{-\beta} \sim \overline{F}_{2Z}(s), && (3.6)\\ \overline{F}_{W^{uc}}(s) &\sim 2^\beta s^{-\beta} + 2^\gamma s^{-\gamma} \sim 2^\beta s^{-\beta}, && (3.7)\\ \textit{and} \qquad \overline{F}_{B^{cs}}(s) &\sim 2^\beta s^{-\beta} + 2 s^{-\gamma}, \quad \overline{F}_{B^{uc}}(s) \sim 2 s^{-\beta} + 2 s^{-\gamma}. && (3.8) \end{aligned}$$

b) For the Value at Risk of the best and of the worst case pairs holds

$$\mathrm{VaR}_\alpha^{cs,ub} \sim \mathrm{VaR}_\alpha^{uc,ub} \sim \frac{2}{(1-\alpha)^{\frac{1}{\beta}}} \tag{3.9}$$

and

$$\mathrm{VaR}_\alpha^{cs,\ell b} \sim \frac{2}{(1-\alpha)^{\frac{1}{\beta}}}, \quad \mathrm{VaR}_\alpha^{uc,\ell b} \sim \frac{2^{\frac{1}{\beta}}}{(1-\alpha)^{\frac{1}{\beta}}}. \tag{3.10}$$

Proof. a) Using a well-known result for the tail of sums of independent subexponentially distributed random variables it holds that

$$\overline{F}_{W^{cs}}(s) \sim 2^\beta s^{-\beta} + 2^\gamma s^{-\gamma} \sim 2^\beta s^{-\beta} \sim \overline{F}_{2Z}(s). \tag{3.11}$$

Similarly,

$$\overline{F}_{W^{uc}}(s) = P(2(Z \oplus \varepsilon_1) \geqslant s) \sim 2^\beta s^{-\beta} + 2^\gamma s^{-\gamma} \sim 2^\beta s^{-\beta}.$$

Thus the difference $\overline{F}_{W^{uc}}(s) - \overline{F}_{W^{cs}}(s)$ is of order $O(s^{-\gamma})$, i.e. $|\overline{F}_{W^{uc}}(s) - \overline{F}_{W^{cs}}| \leqslant C s^{-\gamma}$ for some $C > 0$.

For the best case holds

$$\overline{F}_{B^{cs}}(s) = P(2Z \oplus (\varepsilon_1 + \varepsilon_1^{cm}) \geqslant s) \sim 2^\beta s^{-\beta} + 2s^{-\gamma} \tag{3.12}$$

and

$$\overline{F}_{B^{uc}}(s) = P((Z \oplus \varepsilon_1) + (Z \oplus \varepsilon_1)^{cm} \geqslant s) \sim 2\overline{F}_{Z\oplus\varepsilon_1}(s) \sim 2s^{-\beta} + 2s^{-\gamma}. \tag{3.13}$$

In consequence the improvement of the tail risk is of order $(2^\beta - 2)s^{-\beta}$. So also in this case an essential improvement of the lower bound is observed while the improvement of the upper bound is of minor order.

b) For the Value at Risk of the best and the worst case pairs as consequence of (3.6)–(3.8) the formulas in (3.9) and (3.10) are obtained. □

Remark 3.3. *In the case of exactly Paretian models with* $\overline{F}_Z(s) = s^{-\beta}$ $\overline{F}_{\varepsilon_i}(s) = s^{-\gamma}$, $\gamma > \beta > 1$ *more precisely it holds that*

$$\overline{F}^{uc,ub}(s) - \overline{F}^{cs,ub}(s) \sim 2^\beta s^{-\beta} + 2^\gamma s^{-\gamma} - (2\beta s^{-\beta} + 2s^{-\gamma}) = (2\gamma - 2)s^{-\gamma} \tag{3.14}$$

and

$$\overline{F}^{cs,\ell b}(s) - \overline{F}^{uc,\ell b}(s) \sim 2^\beta s^{-\beta} + 2s^{-\gamma} - 2(s^{-\beta} + s^{-\gamma}) = (2\beta - 2)s^{-\beta}. \tag{3.15}$$

The difference between the upper bounds is of smaller order. In these formulas expansion terms of the independent sum of minor order are neglected.

As a result we obtain in the positive dependent case no improvement of the upper VaR bounds (see (3.10)) and a strong improvement of the lower bound (see (3.11)) by including dependence information by the systematic risk factor Z. The bounds are completely explicit in terms of the tail risk parameter β and the level α and are easy to apply.

3.2 Systemic and idiosyncratic risks of similar magnitude

Next we consider the positive dependent factor case where the systematic risk factor Z and the idiosyncratic risks ε_i are of similar magnitude. As in Section 3.1 Paretian tails are assumed. More precisely, with tail risk $\overline{F}_Z(s) \sim s^{-\beta}$, $\beta > 1$ and $\overline{F}_{\varepsilon_i}(s) \sim \overline{F}_Z(s)$ the following proposition holds:

Proposition 3.4 (Positive dependent risks of similar magnitude)**.**

a)

$$\overline{F}^{cs,ub}(s) \sim (2^\beta + 2)s^{-\beta}, \quad \overline{F}^{uc,ub}(s) \sim 2^{\beta+1}s^{-\beta} \tag{3.16}$$

$$\overline{F}^{cs,\ell b}(s) \sim \frac{s^\beta + 2}{s^\beta}, \quad \overline{F}^{uc.\ell b}(s) \sim \frac{4}{s^\beta} \tag{3.17}$$

b) For the Value at Risk of the best and the worst case pairs are given by

$$\mathrm{VaR}_\alpha^{cs,ub} \sim \frac{(2\beta + 2)^{\frac{1}{\beta}}}{(1-\alpha)^{\frac{1}{\beta}}}, \quad \mathrm{VaR}_\alpha^{uc,ub} \sim \frac{2^{1+\frac{1}{\beta}}}{(1-\alpha)^{\frac{1}{\beta}}}, \tag{3.18}$$

$$\mathrm{VaR}_\alpha^{cs,\ell b} \sim \frac{(2^\beta + 2)^{\frac{1}{\beta}}}{(1-\alpha)^{\frac{1}{\beta}}}, \quad \mathrm{VaR}_\alpha^{uc,\ell b} \sim \frac{4^{\frac{1}{\beta}}}{(1-\alpha)^{\frac{1}{\beta}}}. \tag{3.19}$$

Proof. a) The arguments as in Section 3.1 imply the upper bounds

$$\begin{aligned}\overline{F}^{cs,ub}(s) &= P(2Z \oplus (\varepsilon_1 + \varepsilon_2^c) \geqslant s)\\ &\sim P(2Z \geqslant s) + P(\varepsilon_1 + \varepsilon_2^c \geqslant s)\\ &\sim 2^\beta s^{-\beta} + 2s^{-\beta} = (2^\beta + 2)s^{-\beta}.\end{aligned} \tag{3.20}$$

Similarly, in the unconstrained case holds

$$\overline{F}^{uc,ub}(s) = \overline{F}_{2(Z \oplus \varepsilon_1)}(s) \sim \overline{F}_Z\left(\frac{s}{2}\right) + \overline{F}_{\varepsilon_1}\left(\frac{s}{2}\right) \sim 2^{\beta+1}s^{-\beta}. \tag{3.21}$$

Since for $\beta > 1$, $2^{\beta+1} > 2^\beta + 2$ this implies that the upper risk bound is improved.

For the lower risk bound holds

$$\begin{aligned}\overline{F}^{cs,\ell b}(s) &= P(2Z \oplus (\varepsilon_1 + \varepsilon_1^{cm}) \geqslant s)\\ &\sim P(2Z \geqslant s) + 2P(\varepsilon_1 \geqslant s) \sim \frac{2^\beta}{s^\beta} + \frac{2}{s^\beta} = \frac{2^\beta + 2}{s^\beta}\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\overline{F}^{uc,\ell b}(s) &= P(Z \oplus \varepsilon_1 + (Z \oplus \varepsilon_1)^{cm} \geqslant s)\\ &\sim 2P(Z \oplus \varepsilon_1 \geqslant s) \sim \frac{4}{s^\beta},\end{aligned} \tag{3.23}$$

again a strong improvement of the lower bound.

b) The results in a) imply that the corresponding Value at Risk bounds are given by

$$\mathrm{VaR}_\alpha^{cs,ub} \sim \frac{(2\beta + 2)^{\frac{1}{\beta}}}{(1-\alpha)^{\frac{1}{\beta}}}, \quad \mathrm{VaR}_\alpha^{uc,ub} \sim \frac{2^{1+\frac{1}{\beta}}}{(1-\alpha)^{\frac{1}{\beta}}}. \tag{3.24}$$

Similarly, for the lower bounds it holds that

$$\mathrm{VaR}_\alpha^{cs,\ell b} \sim \frac{(2^\beta + 2)^{\frac{1}{\beta}}}{(1-\alpha)^{\frac{1}{\beta}}}, \quad \mathrm{VaR}_\alpha^{uc,\ell b} \sim \frac{4^{\frac{1}{\beta}}}{(1-\alpha)^{\frac{1}{\beta}}}. \tag{3.25}$$

□

In comparison to the dominant systematic risk case in Section 3.1 we find in the case of risks of similar magnitude increased upper and lower risk bounds due to the non-negligible influence of the idiosyncratic risk factor. Again as in Section 3.1 the improvement of the lower bound by the positive dependence information is more significant.

4 General dependent additive factor models

In the case of negatively dependent additive factor models the opposite effect is observed. The worst case upper bound decreases essentially in the constrained model while the lower bounds remain essentially the same.

We consider a class of additive factor models of the form

$$X_1 = Z \oplus \varepsilon_1, \quad X_2 = Z^{cm} \oplus \varepsilon_2, \tag{4.1}$$

where Z^{cm}, the systematic risk factor for X_2, is the counter monotonic version of Z. Thus in this class the systematic risk factors produce negative dependence. We consider again the case where the systematic risk is dominating:

$$\overline{F}_Z(s) = \overline{F}_{Z^{cm}}(s) \sim s^{-\beta} \text{ and } \overline{F}_{\varepsilon_i}(s) \sim s^{-\gamma}, \quad s \geqslant 1, \ \gamma > \beta > 1. \tag{4.2}$$

The unconstrained risk bounds in this model are the same as those in the positive dependent version in Proposition 3.2. The following result holds:

Proposition 4.1 (Risks in negative dependent, dominant systematic risk case)**.** *For an additive risk factor model $X_1 = Z \oplus \varepsilon_1$, $X_2 = Z^{cm} \oplus \varepsilon_2$ with dominating negatively dependent systematic risk factors as in* (4.2) *the worst case dependence is given by*

$$W^{cs} := (Z + Z^{cm}) \oplus 2\varepsilon_1 \tag{4.3}$$

and the best case is given by

$$B^{cs} := (Z + Z^{cm}) \oplus (\varepsilon_1 + \varepsilon_2^{cm}). \tag{4.4}$$

The corresponding constrained risk bounds are given by

$$\overline{F}^{cs,ub}(s) \sim \frac{2}{s^\beta} + \frac{2^\gamma}{s^\gamma} \sim \frac{2}{s^\beta} \qquad \text{and} \qquad \overline{F}^{cs,\ell b}(s) \sim \frac{2}{s^\beta} + \frac{2}{s^\gamma} \sim \frac{2}{s^\beta}. \tag{4.5}$$

The related Value at Risk bounds are given by

$$\mathrm{VaR}_\alpha^{cs,ub} \sim \frac{2^{\frac{1}{\beta}}}{(1-\alpha)^{\frac{1}{\beta}}} \quad \text{and} \quad \mathrm{VaR}_\alpha^{cs,\ell b} \sim \frac{2^{\frac{1}{\beta}}}{(1-\alpha)^{\frac{1}{\beta}}}. \tag{4.6}$$

Proof. The worst and the best case dependence structures (in convex order) in the constraint model given by (4.1), (4.2) are obtained as in Sections 2 and 3. The corresponding tail risk in (4.5) and the Value at Risk bounds in (4.6) are obtained similarly to the proof of Proposition 3.2. □

As expected the constrained lower bound is a strong improvement of the unconstrained lower bound while the upper bounds are of the same order. The domination of the systematic risk part implies by the negative dependence that upper and lower risk bounds are of the same order.

For the risks in the negatively dependent case with risks of similar magnitude of the risks the following result is obtained:

Proposition 4.2 (Negatively dependent case with risks of similar magnitude)**.** *For the negatively dependent additive factor model in* (4.1) *with risks of similar magnitude i.e. $\overline{F}_Z(s) \sim s^{-\beta}$, $\overline{F}_{\varepsilon_i}(s) \sim s^{-\beta}$, $\beta \geqslant 1$ it holds that*

$$\overline{F}^{cs,ub}(s) \sim \frac{2^\beta + 2}{s^\beta} \quad \text{and} \quad \overline{F}^{cs,\ell b}(s) \sim \frac{4}{s^\beta}. \tag{4.7}$$

The corresponding Value at Risks are given by

$$\mathrm{VaR}_\alpha^{cs,ub} \sim \frac{(2^\beta+2)^{\frac{1}{\beta}}}{(1-\alpha)^{\frac{1}{\beta}}}, \quad \mathrm{VaR}_\alpha^{cs,\ell b} \sim \frac{4^{\frac{1}{\beta}}}{(1-\alpha)^{\frac{1}{\beta}}}. \tag{4.8}$$

As a consequence of the previous results we find for positive dependent systematic risk factors in the dominating and in the case of similar risk magnitude a considerable improvement of the lower risk bounds while the upper risk bounds are of similar order as in the unconstrained case. In the case of negatively dependent systematic risk factors the upper bounds are strongly reduced but the lower bounds are of similar order as in the unconstrained case. In all cases the results are in completely and simple explicit form and are easy to apply.

In the general partially specified additive factor model

$$X_1 = Z_1 \oplus \varepsilon_1, \quad X_2 = Z_2 \oplus \varepsilon_2$$

the dependence structure of the Z_i can vary between these two extremes, i.e.

$$Z_1 + Z_2^{cm} \leqslant_{\mathrm{sm}} Z_1 + Z_2 \leqslant_{\mathrm{sm}} Z_1 + Z_2^c. \tag{4.9}$$

By Theorem 2.6 the worst case and the best case (in convex order) are given by

$$W^{cs} = (Z_1 + Z_2) \oplus 2\varepsilon_1 \tag{4.10}$$

and

$$B^{cs} = (Z_1 + Z_2) \oplus (\varepsilon_1 + \varepsilon_2^{cm}). \tag{4.11}$$

In consequence of (4.9) the tail of the systematic risk parts varies in the Paretian case between the two extremes, the case of positive resp. of negative dependent systematic risk factors, i.e.

$$\frac{2}{s^\beta} \lesssim \overline{F}_{Z_1+Z_2}(s) \lesssim \frac{2^\beta}{s^\beta}. \tag{4.12}$$

The tail risks and corresponding Value at Risks of the joint portfolio, therefore, also vary between the two extremes

$$\frac{2}{s^\beta} + \frac{2}{s^\gamma} \lesssim \overline{F}_{X_1+X_2}(s) \lesssim \frac{2^\beta}{s^\beta} + \frac{2^\gamma}{s^\gamma}. \tag{4.13}$$

Depending on $\beta < \gamma$ or $\beta = \gamma$ (4.13) implies corresponding bounds for the Value at Risk. For an intermediate dependence structure we, therefore, have a reduction of both risk bounds, whose degree depends on the degree of positive resp. negative dependence of the systematic risk factors.

5 Final remarks

In this paper the effect of partially specified additive risk factors on the reduction of risk bounds compared to the risk bounds in unconstrained (marginal) models is analyzed. In an additive class of models $X_1 = Z_1 + \varepsilon_1$, $X_2 = Z_2 + \varepsilon_2$, this can be done in an analytic way. We consider the case where the systematic risk factors Z_i dominate the idiosyncratic risks ε_i as well as the case where both types of risk are of the same order of magnitude. As a result an explanation of the mechanism of risk reduction in such partially specified factor models is obtained.

Systemic risk factors which induce positive dependence help to reduce lower bounds (but not upper bounds). If they induce negative dependence then upper bounds are reduced (but not lower bounds). In intermediate cases of dependence induced by the systematic risk factors one obtains a reduction of both risk bounds whose degree depends on the degree of positive resp. negative dependence of the systematic risk factors. The magnitude of the possible variation of the tail risk is described in formula (4.13). In comparison to the case where the systematic risk factor is dominant one obtains in the case where the systematic and the idiosyncratic risk factors are of similar magnitude increased upper and lower risk bounds due to the influence of the idiosyncratic risk factor.

The results in this paper are given in (nearly) explicit form. The insight obtained in this paper gives for applications of this reduction method a clue to get a qualitative and quantitative impression on the range and on the direction of reduction of the risk bounds due to the incorporation of the dependence properties of the systematic risk factors compared to the unconstrained models.

For general partially specified factor models some qualitative results for this type have been given in Bernard, Rüschendorf, Vanduffel and Wang (2017) which confirm that the behaviour described in this paper in explicit form can also be expected to hold in similar form in general PSFM. The examples given in that paper also demonstrate the great range of reduction of the risk bounds by including this kind of dependence information which is in coincidence with the explicit results for the additive factor models in this paper.

The constrained upper bounds can be determined in a similar way for PSAFM in the general case $n \geqslant 2$ where $X_i = Z_i + \varepsilon_i$, $1 \leqslant i \leqslant n$. For the constrained lower bounds analytic results are not available in full generality for $n \geqslant 3$. As in Bernard, Rüschendorf and Vanduffel (2017) numerical solutions can be given in this general case. For the unconstrained case analytical results on lower convex order bounds for $n \geqslant 3$ are obtained in Bernard et al. (2014) for homogeneous models and in Jakobsons et al. (2016) for inhomogeneous models. These bounds are shown to be sharp under some general assumptions in particular, in the most relevant case of monotone densities (e.g. Pareto). By the arguments as for the proof of Theorem 2.6 these results then lead to sharp convex order lower bounds also in the case $n \geqslant 3$. As a consequence the results in this paper for $n = 2$ concerning convex order hold under general conditions also for $n \geqslant 3$.

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