GENERALIZED STATISTICAL ARBITRAGE CONCEPTS AND RELATED GAIN STRATEGIES

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Abstract. The notion of *statistical arbitrage* introduced in Bondarenko (2003) is generalized to statistical $\mathscr{G}$-arbitrage corresponding to trading strategies which yield positive gains *on average* in a class of scenarios described by a σ-algebra $\mathscr{G}$. This notion contains classical arbitrage as a special case. Admitting general static payoffs as generalized strategies, as done in Kassberger and Liebmann (2017) in the case of one pricing measure, leads to the notion of generalized statistical $\mathscr{G}$-arbitrage. We show that even under standard no-arbitrage there may exist generalized gain strategies yielding positive gains on average under the specified scenarios.

In the first part of the paper we prove that the characterization in Bondarenko (2003), no statistical arbitrage being equivalent to the existence of an equivalent local martingale measure with a path-independent density, is not correct in general. We establish that this equivalence holds true in complete markets and we derive a general sufficient condition for statistical $\mathscr{G}$-arbitrages. As a main result we derive the equivalence of no statistical $\mathscr{G}$-arbitrage to no generalized statistical $\mathscr{G}$-arbitrage.

In the second part of the paper we construct several classes of profitable generalized strategies with respect to various choices of the σ-algebra $\mathscr{G}$. In particular, we consider several forms of embedded binomial strategies and follow-the-trend strategies as well as partition-type strategies. We study and compare their behaviour on simulated data and also evaluate their performance on market data.

1. Introduction

Since the mid-1980s trading strategies which offer profits on average in comparison to little remaining risk have been implemented and analyzed. The starting point were pairs trading strategies, see Gatev et al. (2006) for an historic account and further details. In this strategy one trades two stocks whose prices have a high historic correlation and whose spread widened recently, by buying the loser and shorting the winner. Many variants of this simple strategy followed, see Krauss (2017) and Lazzarino et al. (2018) for surveys and guides to the literature. This raised interest in a deeper theoretical understanding of these approaches.

In this paper, we elaborate and generalize the notion of *statistical arbitrage* (SA) introduced in Bondarenko (2003). The author considers a finite horizon market, represented by the price process of the assets $(S_t)_{t\in[0,T]}$. A trading strategy with zero initial cost is called statistical arbitrage if

(i) the expected payoff is positive and,
(ii) the expected payoff is non-negative *conditional* on S_T.

Unlike pure arbitrage strategies, a statistical arbitrage can have negative payoffs provided the average payoff in each final state is non-negative. This concept supplements previous forms of restrictions like 'good deals' or opportunities with high Sharpe ratios or with high utility (see Hansen and Jagannathan (1991), Cochrane and Saa-Requejo (2000) and Černỳ and Hodges (2002)) or 'approximate arbitrage opportunities' and investment opportunities with a high gain-loss ratio (see Bernardo and Ledoit (2000)). All these restrictions lead to essential reductions of the pricing intervals.

Bondarenko (2003) discusses the concept of statistical arbitrage in connection with various forms of risk preferences, w.r.t. the solution of the joint hypothesis problem, for tests of the efficient market hypothesis (EMH) and the efficient learning market (ELM). The main economic assumption introduced by Bondarenko is the assumption that the pricing kernel is path independent, i.e. it is a function depending only on the final state of the underlying price model but *not* depending on the whole

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history. This assumption implies that the payoff process deflated by the conditional risk neutral density of the final state is a martingale, i.e. has no systematic trend. The main result in (Bondarenko, 2003, Proposition 1) states that the existence of a path-independent pricing kernel is equivalent to the absence of SA strategies.

Hogan et al. (2004) introduce a related approach which considers on an infinite time horizon with trading strategies achieving positive gains on average together with vanishing risk, both in an asymptotic sense. See also Elliott et al. (2005); Avellaneda and Lee (2010).

In Section 2 we generalize the concept of statistical arbitrage. Starting from a σ-field $\mathscr{G}$, a *statistical $\mathscr{G}$-arbitrage* is a trading strategy with positive expected gain, conditional on $\mathscr{G}$. The existence of a pricing measure with $\mathscr{G}$-measurable density implies absence of statistical $\mathscr{G}$-arbitrage. Investigating in detail a class of trinomial models we find that the converse direction in Bondarenko's equivalence theorem is not valid in general. Kassberger and Liebmann (2017) introduced and characterized statistical $\mathscr{G}$-arbitrage w.r.t. generalized (static) strategies in the case where one pricing measure is fixed. In Section 3 we introduce generalized trading strategies including also static or semi-static strategies and derive various characterizations of the corresponding SA concepts for the class of all martingale pricing measures. In Section 4 we fully characterize SA for two-period binomial models and construct statistical arbitrage strategies. These results are used in Section 5 to construct for discrete- and continuous-time models various SA-strategies. We test them in several examples and give an application to market data. A basic class of strategies is obtained by embedding binomial trading strategies into the continuous time models using first-hitting times. Further classes are strategies induced by partitioning the path space and strategies which follow some trend in the data. Several of theses strategies are examined and compared.

2. Statistical $\mathscr{G}$-arbitrage strategies

Consider a filtered probability space $(\Omega, \mathscr{F}, P)$ with a filtration $\mathbb{F} = (\mathscr{F}_t)_{0 \le t \le T}$ and a finite time horizon T. The filtration is assumed to satisfy the usual conditions, i. e. it is right continuous and $\mathscr{F}_0$ contains all null sets of $\mathscr{F}$: if $B \subset A \in \mathscr{F}$ and $P(A) = 0$ then $B \in \mathscr{F}_0$. We also suppose that $\mathscr{F} = \mathscr{F}_T$.

We follow the classical approach to financial markets as for example in Delbaen and Schachermayer (2006). The market itself is given by a $\mathbb{R}^{d+1}$-valued locally bounded semi-martingale $S = (S^0, \dots, S^d)$, i.e. there exists a sequence of stopping times $(T_n)_{n \ge 1}$ tending to ∞ a.s. and a sequence $(K_n)_{n \ge 1}$ of positive constants, such that $|S \mathbb{1}_{[\![0,T_n]\!]}| < K_n$, $n \ge 1$. The numéraire S^0 is set equal to one, such that the prices are considered as already discounted.

A *dynamic trading strategy* ϕ is an S-integrable and predictable process such that the associated value process $V = V(\phi)$ is given by

$$V_t(\phi) = \int_0^t \phi_s \, dS_s, \quad 0 \le t \le T. \tag{1}$$

The trading strategy ϕ is called a-admissible if $\phi_0 = 0$ and $V_t(\phi) \ge -a$ for all $t \ge 0$. ϕ is called *admissible* if it is admissible for some $a > 0$. We further assume that the market is free of arbitrage in the sense of *no free lunch with vanishing risk* (NFLVR), which is equivalent to the existence of an equivalent local martingale measure Q, see Delbaen and Schachermayer (2006). Here, a measure Q which is equivalent to P, $Q \sim P$, such that S is an $\mathbb{F}$-(local) martingale with respect to Q is called equivalent (local) martingale measure, EMM (ELMM). Let $\mathscr{M}^e$ denote the set of all equivalent local martingale measures.

A statistical arbitrage is a dynamic trading strategy which is *on average* profitable, conditional on the final state of the economy S_T. More generally, we consider a general σ-field $\mathscr{G} \subset \mathscr{F}_T$ and consider strategies which are on average profitable conditional on $\mathscr{G}$. For example, $\mathscr{G}$ could be generated by the event $\{S_T > K\}$. We call such strategies $\mathscr{G}$-arbitrage strategies. Sometimes we call a statistical $\mathscr{G}$-arbitrage strategy also a $\mathscr{G}$-profitable strategy or $\mathscr{G}$-arbitrage, for short. By E we denote expectation with respect to the reference measure P.

Definition 2.1. Let $\mathscr{G} \subseteq \mathscr{F}_T$ be a σ-algebra. An admissible dynamic trading strategy ϕ is called a *statistical $\mathscr{G}$-arbitrage strategy*, if

i) $E[V_T(\phi)|\mathscr{G}] \geq 0, \quad P$-a.s.,
ii) $E[V_T(\phi)] > 0$.

Let
$$\mathrm{SA}(\mathscr{G}) := \{\phi : \phi \text{ is a } \mathscr{G}\text{-arbitrage}\}$$
denote the set of all statistical $\mathscr{G}$-arbitrage strategies. The market model satisfies the condition of *no statistical $\mathscr{G}$-arbitrage NSA($\mathscr{G}$)* if
$$\mathrm{SA}(\mathscr{G}) = \emptyset.$$

For $\mathscr{G} = \mathscr{F}_T$, NSA($\mathscr{G}$) is equivalent to the classical no-arbitrage condition (NA) since then $E[V_T(\phi)|\mathscr{G}] = V_T(\phi)$. Recall that NA is implied by NFLVR. If $\mathscr{G} = \sigma(S_T)$, one recovers the notion of statistical arbitrage introduced in Bondarenko (2003) and we use the notation $\mathrm{NSA} = \mathrm{NSA}(\sigma(S_T))$.

A further interesting type of examples is the case where $\mathscr{G} = \sigma(\{S_T \in K_i, i \in \mathcal{I}\})$, $\{K_i\}_{i \in \mathcal{I}}$ being a partition of the state space, such that a statistical arbitrage offers a gain in any $\{S_T \in K_i\}$ *on average*, i.e. $E[V_T(\phi)|S_T \in K_i] \geq 0$ for all $i \in \mathcal{I}$ s.t. $P(S_T \in K_i) > 0$. Similarly one can also consider path-dependent strategies, like for example $\mathscr{G} = \sigma(\{\max_{0 \leq t \leq T} S_t \in K_i, i \in \mathcal{I}\})$.

Remark 2.2. (a) Some immediate consequences of Definition 2.1 are the following:

(i) The tower property of conditional expectations yields that larger σ-fields $\mathscr{G}$ allow for fewer profitable $\mathscr{G}$-arbitrage strategies i. e. $\mathscr{G}_1 \subset \mathscr{G}_2$ implies that $\mathrm{SA}(\mathscr{G}_2) \subset \mathrm{SA}(\mathscr{G}_1)$. As a consequence we get that in this case
$$\mathrm{NSA}(\mathscr{G}_1) \qquad \Rightarrow \qquad \mathrm{NSA}(\mathscr{G}_2). \tag{2}$$
(ii) If $\mathscr{G} = \{\emptyset, \Omega\}$, then $\phi \in \mathrm{SA}(\mathscr{G})$ iff $E_P[V_T(\phi)] > 0$.

(b) The general approach to good-deal bounds in Černỳ and Hodges (2002) allows to consider statistical arbitrages as a special case: indeed, if we define
$$A = \{Z : E[Z|\mathscr{G}] \geq 0 \text{ and } E[Z] > 0\}$$
as set of good deals then a statistical $\mathscr{G}$-arbitrage ϕ is a good-deal strategy if $V_T(\phi) \in A$. The corresponding good-deal pricing bound for an option X is given by
$$\pi(X) = \inf\{x : \exists \phi \text{ admissible s.t. } X + x + V_T(\phi) \in A\},$$
i.e. the smallest price x, such that the portfolio of the option, the price x and the value from a hedging strategy $V_T(\phi)$ is a good deal.

Remark 2.3 (Connection to other concepts of statistical arbitrage)**.** A comprehensive overview of the variety of statistical arbitrage definitions can be found in Lazzarino et al. (2018). This overview shows that in one group of definitions a particular strategy is considered (for example, pairs trading, cointegration strategies, or arbitrage testing strategies) and statistical arbitrage corresponds to the performance in simulations or on real data. A general and far reaching analysis of statistical arbitrage strategies based on the excursion theory of processes (in particular of Markov processes) is given in the recent preprint Ananova et al. (2020).

In comparison to that, the definitions of Bondarenko (2003) and Hogan et al. (2004) and their generalizations are more conceptual. They are not restricted to particular strategies, but ask the question: is there a general trading strategy available implying arbitrage in the sense of positive conditional expectation.

These two concepts can be linked as follows: note that the definition of Bondarenko considers a finite time horizon. Iterating this strategy over time under some kind of stationarity or mean reversion, one gets as a consequence a statistical arbitrage strategy in the asymptotic sense of Hogan et al. The main technical tool to achieve this are boundary crossing probabilities and in particular the results from excursion theory as in Ananova et al. From this viewpoint, the definition of statistical arbitrage as in the (weak) sense of Bondarenko is also of interest for the asymptotic definition in Hogan et al.

Also for a finite time horizon statistical arbitrage can be realized by repetition making use of a law of large numbers. We use this principle in examples in Section 5 of this paper. There we more generally construct profitable strategies in the sense of measuring statistical arbitrage with respect to several classes of σ-fields.

The connection of the statistical arbitrage notions of Bondarenko and Hogan et al. to the above mentioned group of definitions focussing on particular strategies (pairs trading, cointegration, etc., as in Lazzarino et al. (2018) or Avellaneda and Lee (2010)), is as follows: in these works several strategies are compared on the basis of the realized Sharpe ratio. This is in agreement with the asymptotic viewpoint in Hogan et al. who require, in economic terms, that a statistical arbitrage opportunity produces riskless incremental profit with an persisting positive Sharpe ratio in the limit. Therefore, also the Bondarenko notion, even if seemingly formulated in a weak sense, can be seen in close connection with the other more specific statistical arbitrage notions in the literature and therefore, is also well motivated from a practical point of view. ◇

Proposition 1 in Bondarenko (2003) states that (in discrete time), NSA is equivalent to the existence of an equivalent martingale measure Q with path independent density Z, i. e.

$$\frac{dQ}{dP} = Z \in \sigma(S_T), \tag{3}$$

where we use the notation $Z \in \sigma(S_T)$ for Z being $\sigma(S_T)$-measurable. We establish in Section 2.2, that this equivalence is incorrect in general. However, in Section 3 we show that this equivalence holds if the market is complete. We also establish that the statistical no $\mathscr{G}$-arbitrage NSA($\mathscr{G}$)-condition is equivalent to the corresponding no-$\mathscr{G}$-arbitrage condition w.r.t. generalized strategies. In Sections 4 and 5 we explicitly construct statistical arbitrages.

On the other side, existence of an equivalent martingale measure with path independent density Z implies that NSA holds without further assumptions. This also holds true for the generalized notion NSA($\mathscr{G}$), as we show next.

Theorem 2.4. *If there exists $Q \in \mathscr{M}^e$ such that $\frac{dQ}{dP}$ is $\mathscr{G}$-measurable, then NSA($\mathscr{G}$) holds.*

Proof. The proof follows from the Bayes-formula for conditional expectations. We denote by $\underline{L}_b = \underline{L}_b(P)$ the set of random variables bounded P-almost surely from below. If $Z = \frac{dQ}{dP} \in \mathscr{G}$, then for any $X \in \underline{L}_b$ it holds that

$$E_Q[X \,|\, \mathscr{G}] = \frac{E_P[XZ \,|\, \mathscr{G}]}{E_P[Z \,|\, \mathscr{G}]} = E_P[X \,|\, \mathscr{G}]. \tag{4}$$

If there would be a statistical arbitrage strategy ϕ with $E_P[X \,|\, \mathscr{G}] \geq 0$ and $E_P[X] > 0$, where $X = V_T(\phi)$, then, by (4),

$$E_Q[X \,|\, \mathscr{G}] \geq 0, \qquad Q\text{-a.s.}$$

Moreover, since ϕ is admissible, $V(\phi)$ is a Q-supermartingale by Fatou's lemma, and we obtain that

$$E_Q[X] = E_Q[V_T(\phi)] \leq V_0(\phi) = 0. \tag{5}$$

Hence,

$$0 = E_Q[X \,|\, \mathscr{G}] = E_P[X \,|\, \mathscr{G}]$$

in contradiction to $E_P[X] > 0$. □

Remark 2.5 (Alternative admissible strategies)**.** An inspection of the proof, in particular Equation (5), shows that the claim also holds when we consider as admissible such strategies ϕ for which $V(\phi)$ is a Q-supermartingale.

In the following we establish that the converse direction in the Bondarenko result is not true. For the construction of a counterexample we characterize next statistical arbitrage in a certain class of trinomial models.

2.1. **Statistical arbitrage in trinomial models.** In this subsection we consider a one-dimensional trinomial model of the following type which we call *the trinomial model*. While the first step is binomial, the second time-step is trinomial. Assume that $d = 1$, $\Omega = \{\omega_1, \dots, \omega_6\}$ and $T = 2$. Let $S_0 = s_0 \in \mathbb{R}_{\geq 0}$ and let S_1 and S_2 be defined by

$$\begin{aligned}
&S_1(\omega_1) = S_1(\omega_2) = S_1(\omega_3) = s_1^+, \quad S_1(\omega_4) = S_1(\omega_5) = S_1(\omega_6) = s_1^-. \\
&S_2(\omega_2) = s_2^{++}, \qquad S_2(\omega_3) = S_2(\omega_5) = s_2^{+-}, \qquad S_2(\omega_6) = s_2^{--}, \\
&S_2(\omega_1) = S_2(\omega_4) = s_2^\circ;
\end{aligned}$$

$s_1^+ - s_0 > 0$, $s_1^- - s_0 < 0$ together with $s_2^\circ > s_2^{++} > s_2^{+-} > s_2^{--} > 0$, see Figure 1.

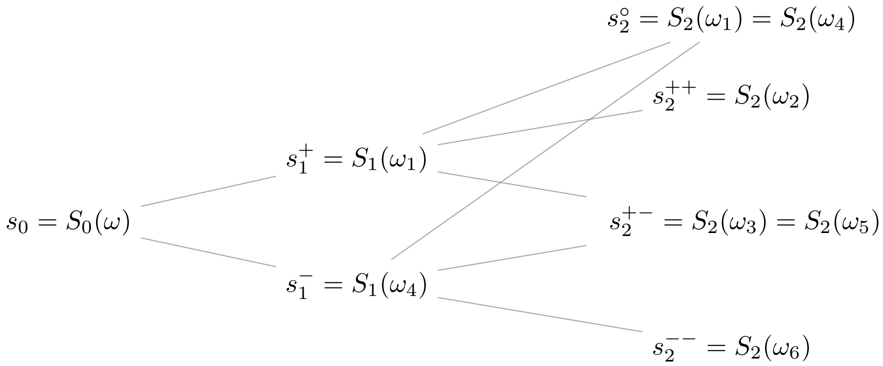

FIGURE 1. The considered trinomial model with $T = 2$ time steps. The first step is binomial, the second step is also (recombining) binomial with an additional top state $\{\omega_1, \omega_4\}$.

The existence of an equivalent martingale measure is equivalent to $\Delta S_i = S_i - S_{i-1}$ taking positive as well as negative values in each sub-tree and we assume that $s_2^{++} - s_1^+ > 0$, $s_1^- < s_2^{+-} < s_1^+$, $s_2^{--} - s_1^- < 0$. The gains from trading with a self-financing strategy $\phi = (\phi_1, \phi_2)$ are given by

$$V_2(\phi) = \phi_1 \Delta S_1 + \phi_2 \Delta S_2. \tag{6}$$

While ϕ_1 is constant since $\mathscr{F}_0 = \{\emptyset, \Omega\}$, ϕ_2 can take two different values which we denote by ϕ_2^+ and ϕ_2^- (taken in the states $\{\omega_1, \omega_2, \omega_3\}$ and $\{\omega_4, \omega_5, \omega_6\}$, respectively). With $\mathscr{G} = \sigma(S_2) = \sigma(\{\omega_1, \omega_4\}, \{\omega_3, \omega_5\}, \{\omega_2\}, \{\omega_6\})$ the strategy ϕ is a statistical arbitrage if and only if

$$\begin{aligned}
&\phi_1 \Delta S_1(\omega_2) + \phi_2^+ \Delta S_2(\omega_2) \geq 0, \\
&\phi_1 \Delta S_1(\omega_6) + \phi_2^- \Delta S_2(\omega_6) \geq 0, \\
&\phi_1 \Delta S_1(\omega_1) P(\omega_1) + \phi_2^+ \Delta S_2(\omega_1) P(\omega_1) + \phi_1 \Delta S_1(\omega_4) P(\omega_4) + \phi_2^- \Delta S_2(\omega_4) P(\omega_4) \geq 0, \\
&\phi_1 \Delta S_1(\omega_3) P(\omega_3) + \phi_2^+ \Delta S_2(\omega_3) P(\omega_3) + \phi_1 \Delta S_1(\omega_5) P(\omega_5) + \phi_2^- \Delta S_2(\omega_5) P(\omega_5) \geq 0,
\end{aligned} \tag{7}$$

and, in addition, at least one of the inequalities is strict.

Moreover, if we consider an equivalent martingale measure Q then the density Z is path-independent if and only if $Z(\omega_1) = Z(\omega_4)$ and $Z(\omega_3) = Z(\omega_5)$. In order to establish a criterion for our model to be free of statistical arbitrage, denote

$$\begin{aligned}
\Gamma_1 &= \frac{-\Delta S_1(\omega_5) + \Delta S_2(\omega_5) \frac{\Delta S_1(\omega_6)}{\Delta S_2(\omega_6)}}{\Delta S_1(\omega_3) - \Delta S_2(\omega_3) \frac{\Delta S_1(\omega_2)}{\Delta S_2(\omega_2)}}, \\
\Gamma_2 &= \frac{\frac{\Delta S_1(\omega_6)}{\Delta S_2(\omega_6)} (\Delta S_2(\omega_4) + \Delta S_2(\omega_5)) - \Delta S_1(\omega_4) - \Delta S_1(\omega_5)}{\Delta S_1(\omega_3) - \Delta S_1(\omega_1) \frac{\Delta S_2(\omega_3)}{\Delta S_2(\omega_1)}}.
\end{aligned}$$

Lemma 2.6. *Let $\nu_1 := \frac{P(\omega_1)}{P(\omega_4)}$, $\nu_2 := \frac{P(\omega_3)}{P(\omega_5)}$. In the trinomial model there is no statistical arbitrage, if*

(i) $\nu_1 = -\frac{\Delta S_2(\omega_3)}{\Delta S_2(\omega_1)} \nu_2$, *and*
(ii) $\Gamma_1 < \nu_2 \leq \Gamma_2$.

The proof is relegated to the appendix.

2.2. **A counter example.** In the following we use Lemma 2.6 to show that the equivalence result in Proposition 1 in Bondarenko (2003) is not valid in general. Consider the incomplete trinomial model with

$$(s_0, s_1^+, s_1^-, s_2^{++}, s_2^{+-}, s_2^{--}, s_2^\circ) = (10, 12, 8, 13, 10, 6, 14).$$

It is easy to check that the equivalent martingale measures Q specified by $q = (Q(\omega_1), \dots, Q(\omega_6))$ are given by the set

$$\mathscr{Q} = \left\{ q \in \mathbb{R}^6 \,\middle|\, q_1 = -\frac{3}{4}q_2 + \frac{1}{4}, q_3 = -\frac{1}{4}q_2 + \frac{1}{4}, q_4 = q_6 - \frac{1}{4}, q_5 = -2q_6 + \frac{3}{4} : q_2 \in \left(0, \frac{1}{3}\right), q_6 \in \left(\frac{1}{4}, \frac{3}{8}\right) \right\}.$$

Furthermore, the underlying measure P is specified by the vector $p = (P(\omega_1), \dots, P(\omega_6))$ with

$$p = (0.15, 0.2, 0.3, 0.05, 0.1, 0.2).$$

We compute $\nu_1 = \frac{p_1}{p_4} = 3$ and $\nu_2 = \frac{p_3}{p_5} = 3$. Then

$$\Gamma_2 = \frac{\frac{\Delta S_1(\omega_6)}{\Delta S_2(\omega_6)}(\Delta S_2(\omega_4) + \Delta S_2(\omega_5)) - \Delta S_1(\omega_4) - \Delta S_1(\omega_5)}{\Delta S_1(\omega_3) - \Delta S_1(\omega_1)\frac{\Delta S_2(\omega_3)}{\Delta S_2(\omega_1)}} = 3 = \nu_2,$$

$$\Gamma_1 = \frac{-\Delta S_1(\omega_5) + \Delta S_2(\omega_5)\frac{\Delta S_1(\omega_6)}{\Delta S_2(\omega_6)}}{\Delta S_1(\omega_3) - \Delta S_2(\omega_3)\frac{\Delta S_1(\omega_2)}{\Delta S_2(\omega_2)}} = \frac{2}{3} < \nu_2$$

and

$$\nu_1 = -\frac{\Delta S_2(\omega_3)}{\Delta S_2(\omega_1)}\nu_2 = \nu_2 = 3 = \frac{p_1}{p_4}.$$

According to Lemma 2.6 there is no statistical arbitrage in the stated example. But, on the other hand, there is no path independent density in this case because if there would be a path independent density, i. e. a density Z with $Z(\omega_1) = Z(\omega_4)$ and $Z(\omega_3) = Z(\omega_5)$, there would exist an equivalent martingale measure Q fulfilling the conditions

$$\frac{q_1}{q_4} = \frac{p_1}{p_4} = 3 \qquad \text{and} \qquad \frac{q_3}{q_5} = \frac{p_3}{p_5} = 3. \tag{8}$$

But the only $q \geq 0$ fulfilling (8) is $q = (\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{3})$ which is not an element of the set $\mathscr{Q}$ of equivalent martingale measures.

This example shows that the converse of Theorem 2.4 does not hold in general and therefore, Proposition 1 in Bondarenko (2003) needs additional assumptions: indeed, we have shown that there does not exist a statistical arbitrage and at the same time there is no path-independent density to an equivalent martingale measure.

3. Generalized statistical $\mathscr{G}$-arbitrage strategies

For risk and investment optimization problems it has been shown that conditioning of payoffs on the pricing density leads to improved (cost efficient) payoffs (see Burgert and Rüschendorf (2006), Kassberger and Liebmann (2017) and several papers cited therein). In connection with these improvement procedures, Kassberger and Liebmann (2017) introduced, in the case where one pricing measure Q is specified, the followingg notion of generalized statistical $\mathscr{G}$-arbitrage for a general static payoff X considered as a generalized strategy. Note that in their context no financial market is specified, Q is any probability measure dominated by P. When a concrete financial market is considered, only martingale pricing measures Q will be considered.

Definition 3.1. Let $\mathscr{G} \subseteq \mathscr{F}$ be a σ-algebra. The set of *generalized statistical $\mathscr{G}$-arbitrage-strategies* with respect to a pricing measure $Q \ll P$ is defined as

$$\overline{\mathrm{SA}}(Q, \mathscr{G}) := \{X \in \underline{L}_b : E_Q[X] \leq 0,\ E_P[X|\mathscr{G}] \geq 0 \ P\text{-a.s. and } E_P[X] > 0\}.$$

The market satisfies $\overline{\mathrm{NSA}}(Q, \mathscr{G})$, the condition of *no generalized statistical $\mathscr{G}$-arbitrage* with respect to Q, if

$$\overline{\mathrm{SA}}(Q, \mathscr{G}) = \emptyset.$$

The following result in Kassberger and Liebmann (2017), Proposition 6, characterizes the generalized $\overline{\mathrm{NSA}}(Q, \mathscr{G})$-condition by showing that this notion is equivalent to $\mathscr{G}$-measurability of $dZ = \frac{dQ}{dP}$.

Proposition 3.2. *Let $Q \sim P$ be an equivalent pricing measure. Then, the no-arbitrage condition $\overline{NSA}(Q, \mathscr{G})$is equivalent to the existence of a $\mathscr{G}$-measurable version of the Radon-Nikodym derivative $Z = \frac{dQ}{dP}$.*

The proof of this result is achieved by Jensen's inequality and using as candidate of a generalized $\mathscr{G}$-arbitrage

$$X = \frac{E_P[Z \,|\, \mathscr{G}]}{Z} - 1 \geq -1. \tag{9}$$

We aim at studying under which conditions there exist generalized statistical $\mathscr{G}$-arbitrages and to describe connections between $\overline{\text{NSA}}(Q, \mathscr{G})$ and $\text{NSA}(\mathscr{G})$. One consequence of Proposition 3.2 is the characterization of $\text{NSA}(\mathscr{G})$ for the case of complete market models. Recall that the Radon-Nikodym derivative $Z = \frac{dQ}{dP}$ is path-independent, iff Z is $\sigma(S_T)$-measurable.

A financial market is called *complete*, if every contingent claim is attainable, i.e. for every $\mathscr{F}$-measurable random variable X bounded from below, we find an admissible self-financing trading strategy ϕ, such that $x + V_T(\phi) = X$. This is implied by the assumption that $\mathscr{M}^e = \{Q\}$: indeed, under this assumption, Theorem 16 in Delbaen and Schachermayer (1995) yields that any X, bounded from below, is hedgeable and hence attainable.

Theorem 3.3. *Assume that $\mathscr{M}^e = \{Q\}$. Then NSA($\mathscr{G}$) holds if and only if $\frac{dQ}{dP}$ is $\mathscr{G}$-measurable.*

Proof. If $Q \in \mathscr{M}^e$ has a $\mathscr{G}$-measurable density $\frac{dQ}{dP}$, then, by Theorem 2.4, $\text{NSA}(\mathscr{G})$ holds.

For the converse direction assume that Z is not $\mathscr{G}$-measurable. By Proposition 3.2 it follows that there exists a generalized $\mathscr{G}$-arbitrage, i.e. an $X \in \underline{L}_b$ with $E_Q[X] \leq 0$, $E_P[X \,|\, \mathscr{G}] \geq 0$ and $E_P[X] > 0$. Hence, Theorem 16 in Delbaen and Schachermayer (1995) yields existence of an admissible self-financing trading strategy ϕ, such that $x + V_T(\phi) = X$. Moreover, the superhedging duality, i.e. Theorem 9 in Delbaen and Schachermayer (1995) implies that $x = E_Q[X] = 0$, and hence ϕ is a $\mathscr{G}$-arbitrage. This is a contradiction and the claim follows. □

In particular this result implies that equivalence result (Bondarenko, 2003, Proposition 1) gives a correct characterization of NSA for complete markets. In the following example we give a class of diffusion processes for which, as consequence of Proposition 3.2 and Theorem 3.3, no statistical $\mathscr{G}$-arbitrage for $\mathscr{G} = \sigma(S_T)$ exists.

Example 3.4 (Statistical arbitrage for diffusions)**.** This example discusses the consequences of Proposition 3.2 and Theorem 3.3 in the case of a diffusion model. Consider a P-Brownian motion B generating the filtration $\mathbb{F} = \mathbb{F}^B$. Let S be a one-dimensional diffusion process satisfying $S_0 = s_0$, $s_0 \in \mathbb{R}$ and

$$dS_t = S_t a_t \, dt + S_t b_t \, dB_t, \quad 0 \leq t \leq T, \tag{10}$$

a and b deterministic and sufficiently integrable. Denote the market price of risk by $\lambda_t = a_t / b_t$. Then this model is complete and by Girsanov's theorem has a unique equivalent local martingale measure Q with Radon-Nikodym derivative

$$Z_T = \exp\left(-\int_0^T \lambda \, dB_t - \frac{1}{2} \int_0^T \lambda_t^2 \, dt \right). \tag{11}$$

This illustrates that in general, Z_T will not be a function of S_T. However, if $\lambda = c \cdot b$, $c \in \mathbb{R}$, then this can be possible. Indeed, in this case

$$Z_T = \exp\left(-c \int_0^T b_t dB_t - \int_0^T \frac{a_t^2}{2b_t^2} dt \right)$$

is a function of S_T. We obtain from Theorem 3.3 that there are no statistical arbitrage opportunities with respect to $\mathscr{G} = \sigma(S_T)$. This holds in particular when $a_t = a_0$ and $b_t = b_0$, $0 \leq t \leq T$, i.e. in the case of constant drift and volatility (the Black-Scholes model). On the other side, the diffusion model allows for statistical arbitrage in general. A comparable result was obtained in Göncü (2015) when studying the concept of statistical arbitrage introduced in Hogan et al. (2004).

The following definition extends the notion of generalized statistical $\mathscr{G}$-arbitrage with respect to a single pricing measure Q in Definition 3.1 to the consideration of a class $\mathscr{Q}$ of pricing measures. As a main result of this paper we obtain that in the case $\mathscr{Q} = \mathscr{M}^e$ the corresponding no-arbitrage condition w.r.t. generalized strategies is equivalent to the no-arbitrage condition $\text{NSA}(\mathscr{G})$ w.r.t. trading strategies.

Definition 3.5. Let $\mathscr{G} \subseteq \mathscr{F}$ be a σ-algebra. The set of *generalized statistical $\mathscr{G}$-arbitrage-strategies* is defined as

$$\overline{\mathrm{SA}}(\mathscr{G}) := \left\{ X \in \underline{L}_b : \sup_{Q \in \mathscr{M}^e} E_Q[X] \leq 0,\ E_P[X|\mathscr{G}] \geq 0\ P\text{-a.s. and } E_P[X] > 0 \right\}.$$

The market satisfies $\overline{\mathrm{NSA}}(\mathscr{G})$, i.e. *no generalized statistical $\mathscr{G}$-arbitrage*, if

$$\overline{\mathrm{SA}}(\mathscr{G}) = \emptyset.$$

To establish a connection between statistical $\mathscr{G}$-arbitrage trading strategies and generalized statistical $\mathscr{G}$-arbitrage strategies, we recall that we assumed use the concept of *No Free Lunch with Vanishing Risk* (NFLVR). Recall that the underliyng process S is assumed to satisfy (NFLVR).

Theorem 3.6. *On the financial market given by S it holds that*

$$SA(\mathscr{G}) = \overline{SA}(\mathscr{G}),$$

and, in particular,

$$\overline{NSA}(\mathscr{G}) \Leftrightarrow NSA(\mathscr{G}).$$

Proof. We first show that every $\mathscr{G}$-arbitrage strategy is a generalized $\mathscr{G}$-arbitrage strategy: consider $\phi \in \mathrm{SA}(\mathscr{G})$, i.e. $E[V_T(\phi)\,|\,\mathscr{G}] \geq 0$ and $E[V_T(\phi)] > 0$. By the superreplication duality, Theorem 9 in Delbaen and Schachermayer (1995), it holds that

$$\sup_{Q \in \mathscr{M}^e} E_Q[V_T(\phi)] = \inf\{x \,|\, \exists \text{ admissible } \tilde{\phi}, x + V_T(\tilde{\phi}) \geq V_T(\phi)\}.$$

Choosing $\tilde{\phi} = \phi$ it follows $\sup_{Q \in \mathscr{M}^e} E_Q V_T(\phi) \leq 0$. Note that in addition, admissibility of ϕ implies that $V_T(\phi)$ is bounded from below and so $V_T(\phi) \in \overline{SA}(\mathscr{G})$.

For the reverse implication we have, again by the superreplication duality, for $X \in \overline{SA}(\mathscr{G})$ that

$$0 \geq \sup_{Q \in \mathscr{M}^e} E_Q X = \inf\{x \in \mathbb{R} \,|\, \exists \text{ admissible } \phi,\ x + V_T(\phi) \geq X\}.$$

Since the infimum is finite, Theorem 9 in Delbaen and Schachermayer (1995) yields that it is indeed a minimum. Without loss of generality, we may chose $x = 0$ and obtain the existence of an admissible dynamic trading strategy ϕ with $X \leq V_T(\phi)$. As $X \in \overline{SA}(\mathscr{G})$ it holds further that $E_P[X\,|\,\mathscr{G}] \geq 0$, P-a.s., which leads us to

$$E_P[V_T(\phi)\,|\,\mathscr{G}] \geq E_P[X\,|\,\mathscr{G}] \geq 0 \quad P\text{-a.s.}$$

Then, $E_P[V_T(\phi)] \geq E_P[X] > 0$, such that $V_T(\phi) \in \mathrm{SA}(\mathscr{G})$. So the existence of generalized $\mathscr{G}$-arbitrage strategies is equivalent to the existence of $\mathscr{G}$-arbitrage strategies $V_T(\phi)$ in $\mathrm{SA}(\mathscr{G})$and the claim follows. □

Using Theorem 2.4, we obtain from Theorem 3.6, that existence of an equivalent local martingale measure with a $\mathscr{G}$-measurable density implies even the absence of generalized $\mathscr{G}$-statistical arbitrages.

Corollary 3.7. *If there exists $Q \in \mathscr{M}^e$ such that $\frac{dQ}{dP}$ is $\mathscr{G}$-measurable, then $\overline{NSA}(\mathscr{G})$ holds.*

We remark that the converse of this corollay is still open in the general case.

4. Statistical arbitrage strategies in binomial models

In this section we propose a method to construct trading strategies in binomial models yielding statistical arbitrages. These strategies will be used in the Section 5 for the construction of profitable trading strategies in continuous time by embedding binomial models.

Consider the recombining two-period binomial model: assume that $\Omega = \{\omega_1, \ldots, \omega_4\}$ and $T = 2$. Let $S_0 = s_0 > 0$ and let $S_1(\omega_1) = S_1(\omega_2) = s^+$, and $S_1(\omega_3) = S_1(\omega_4) = s^-$ as well as $s^{++} = S_2(\omega_1)$, $s^{+-} = S_2(\omega_2) = S_2(\omega_3)$, and $s^{--} = S_2(\omega_4)$. Absence of arbitrage is equivalent to ΔS_i, $i = 1, 2$ taking positive as well as negative values. We assume without loss of generality that $s^+ > s_0$, $s^- < s_0$, and

$s^{++} > s^+$, $s^- < s^{+-} < s^+$, and $s^{--} < s^-$. Gains from trading are again given by (6). Moreover, ϕ_1 is constant and ϕ_2 can take the two values $\{\phi_2^+, \phi_2^-\}$. Then $\phi = (\phi_1, \phi_2)$ is a statistical arbitrage, iff

$$\begin{aligned}
&\phi_1 \Delta S_1(\omega_1) + \phi_2^+ \Delta S_2(\omega_1) \geq 0 \\
&\phi_1 \Delta S_1(\omega_4) + \phi_2^- \Delta S_2(\omega_4) \geq 0 \\
&\phi_1 \Delta S_1(\omega_2) P(\omega_2) + \phi_2^+ \Delta S_2(\omega_2) P(\omega_2) + \phi_1 \Delta S_1(\omega_3) P(\omega_3) + \phi_2^- \Delta S_2(\omega_3) P(\omega_3) \geq 0
\end{aligned} \tag{12}$$

and at least one of the inequalities is strict. The density Z is path-independent if and only if $Z(\omega_2) = Z(\omega_3)$. Equations (12) are equivalent to $A\boldsymbol{\phi} \geq 0$, $\boldsymbol{\phi} = (\phi_1, \phi_2^+, \phi_2^-)^\top$ with

$$A = \begin{pmatrix} \Delta S_1(\omega_1) & \Delta S_2(\omega_1) & 0 \\ \Delta S_1(\omega_4) & 0 & \Delta S_2(\omega_4) \\ q\Delta S_1(\omega_2) + \Delta S_1(\omega_3) & q\Delta S_2(\omega_2) & \Delta S_2(\omega_3) \end{pmatrix}, \tag{13}$$

where $q = \frac{P(\omega_2)}{P(\omega_3)}$.

Proposition 4.1. *In the recombining two-period binomial model NSA holds if and only if* $\det(A) = 0$. *Moreover,* $\det(A) = 0$ *is equivalent to*

$$\frac{P(\omega_2)}{P(\omega_3)} = \frac{\Delta S_2(\omega_1)(\Delta S_1(\omega_3)\Delta S_2(\omega_4) - \Delta S_1(\omega_4)\Delta S_2(\omega_3))}{\Delta S_2(\omega_4)(\Delta S_1(\omega_1)\Delta S_2(\omega_2) - \Delta S_1(\omega_2)\Delta S_2(\omega_1))} =: \tilde{q}. \tag{14}$$

.

The proof is relegated to the appendix.

Lemma 4.2. *Consider the recombining two-period binomial model with statistical arbitrage. In this model,* $\boldsymbol{\phi} = \frac{1}{D}(\xi^1, \xi^2, \xi^3)$ *with*

$$\begin{aligned}
\xi^1 &= \big(q\Delta S_2(\omega_2) - \Delta S_2(\omega_1)\big)\Delta S_2(\omega_4) + \Delta S_2(\omega_1)\Delta S_2(\omega_3), \\
\xi^2 &= -\big(\Delta S_1(\omega_3) + q\Delta S_1(\omega_2) - \Delta S_1(\omega_1)\big)\Delta S_2(\omega_4) - \big(\Delta S_1(\omega_1) - \Delta S_1(\omega_4)\big)\Delta S_2(\omega_3), \\
\xi^3 &= -\big(q\Delta S_1(\omega_4) - q\Delta S_1(\omega_1)\big)\Delta S_2(\omega_2) - \big(-\Delta S_1(\omega_4) + \Delta S_1(\omega_3) + q\Delta S_1(\omega_2)\big)\Delta S_2(\omega_1),
\end{aligned}$$

$q = \frac{P(\omega_2)}{P(\omega_3)}$, *and*

$$D = \Big(q\Delta S_1(\omega_1)\Delta S_2(\omega_2) + \big(-\Delta S_1(\omega_3) - q\Delta S_1(\omega_2)\big)\Delta S_2(\omega_1)\Big)\Delta S_2(\omega_4) + \Delta S_1(\omega_4)\Delta S_2(\omega_1)\Delta S_2(\omega_3)$$

is a statistical arbitrage.

Proof. If $\frac{P(\omega_2)}{P(\omega_3)} \neq \tilde{q}$ we have statistical arbitrage according to Proposition 4.1 and that the determinant of the matrix A in (13) is not equal to zero. In this case the matrix A is invertible. Hence, $\boldsymbol{\phi} = A^{-1}\mathbb{1}$ is a statistical arbitrage and it is easily verified that $\boldsymbol{\phi} = \frac{1}{D}(\xi^1, \xi^2, \xi^3)$. ☐

Remark 4.3 (Risk of statistical arbitrages)**.** The word *arbitrage* might be misleading on the riskiness of statistical arbitrages, because in the classical sense, an arbitrage is a strategy without risk. This is of course *not* the case for statistical arbitrages (or the following generalizations of this concept). Since we consider arbitrage-free markets, all gains come with a certain risk and, higher profits are associated with higher risk. This is confirmed by our simulation results in the following section.

As a simple example consider the case of the binomial model where $\Delta_i S(\omega_j) \in \{5, -5\}$, i.e. the stock either rises by 5 or falls by 5. In addition, assume that $q = P(\omega_2)/P(\omega_3) = 1.2$. Then, using Equation (13) it is not difficult to compute $\boldsymbol{\phi} = A^{-1}\mathbf{1} = (1.6, -1.4, -1.8)^\top$. From this strategy we obtain that the gains at time 2, given by

$$G_2(\omega) = \phi_1(\omega)\Delta S_1(\omega) + \phi_2(\omega)\Delta S_2(\omega),$$

yield $G_2(\omega_1) = G_2(\omega_4) = 1$, corresponding to (12). In addition, we obtain that $G_2(\omega_2) = 15$ and $G_2(\omega_3) = -17$. If we assume that $P(\omega_2) = 0.3$ we obtain that the average expected gain on $\{\omega_2, \omega_3\}$ computes to

$$P(\omega_2)G_2(\omega_2) + P(\omega_3)G_3(\omega_3) = 0.3 \cdot 15 + 0.25 \cdot (-17) = 0.25 \geq 0, \tag{15}$$

such that the strategy is indeed a statistical arbitrage. While the (average) gains in the three relevant scenarios are $1, 0.25, 1$, the possible loss in scenario ω_3 is equal to -17, which is attained with probability 0.25, clearly pointing out the riskiness of the strategy.

To exploit the averaging property of *statistical* arbitrage, we keep repeating this strategy until we first record a positive P&L. These considerations show clearly, that a risk analysis of the implemented strategy is important.

5. Profitable strategies

Up to now we saw conditions and examples of statistical arbitrages in a variety of models. Here we are considering several classes of simple statistical arbitrage strategies for several classes of σ-fields $\mathscr{G}$. While these strategies are useful and easy to apply for general stochastic models we investigate them on the Black-Scholes model which allows to explicitly check by analytic formulas for the involved stopping times the corresponding (no-)arbitrage condition. For more general models this has to be checked numerically.

The Black-Scholes model is, according to Example 3.4, free of statistical arbitrage. For some choices of $\mathscr{G}$, however, $\mathscr{G}$-arbitrage strategies may exist in the Black-Scholes model. We show in the following how to construct dynamic trading strategies allowing statistical $\mathscr{G}$-arbitrage for various choices of $\mathscr{G}$. To this end, assume that S is a geometric Brownian motion, i.e. the unique strong solution of the stochastic differential equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t, \qquad 0 \le t \le T \tag{16}$$

where B is a P-Brownian motion and $\sigma > 0$. In the simulation we will first chose $\mu = 0.1241$, $\sigma = 0.0837$, $S_0 = 2186$ according to estimated drift and volatility from the S&P 500 (September 2016 to August 2017), and later consider small variations.

Motivated by our findings in Section 2.1, we begin by embedding binomial trading strategies into the diffusion setting by considering two limits (up / down) and taking actions at the first times these limits are reached. In Section 5.3 we will introduce some related follow-the-trend strategies.

5.1. Embedded binomial trading strategies.

We introduce a recombination of several two-step binomial models embedded in the continuous-time model as long as the final time T is reached. We consider the σ-fields generated by the stopping times when the final states of each of the binomial models are reached (or the trivial σ-field otherwise).

As we repeatedly consider embedded binomial models it makes sense to consider the outcome of the trading strategy *on average* conditional on the final states of each binomial model, i.e. by averaging the outcome over many repeated applications of the trading strategy and hence we get in this way an estimate for the *statistical arbitrage* in the whole time interval $[0, T]$.

Let i denote the current step of our iteration and consider a multiplicative step size $c > 0$. We initialize at time $t_0^0 = 0$. Otherwise consider the initial time of our next iteration given by the time when the last repetition finished and denote this time by t_0^i and the according level by $s_0^i = S_{t_0^i}$. Then we define the following two stopping times corresponding to the first and second period of our binomial model by

$$t_1^i = \inf \big\{ t \in [t_0^i, T] \,|\, S_t \in \{ s_0^i(1-c), s_0^i(1+c) \} \big\} \tag{17}$$

and

$$t_2^i = \inf \big\{ t \in (t_1^i, T] \,|\, S_t \in \{ s_0^i(1-2c), s_0^i, s_0^i(1+2c) \} \big\}, \tag{18}$$

with the convention that $\inf \emptyset = T$. This induces a sequence of σ-fields

$$\mathscr{G}^i := \sigma(S_{t_2^i}).$$

Since S is continuous, this scheme allows to embed repeated binomial models $S_{t_0^i}$, $S_{t_1^i}$, $S_{t_2^i}$, $i = 1, 2, \ldots, N$ into continuous time with a random number N of repetitions. The proposed trading strategy then is to execute successively the statistical arbitrage strategy for binomial models computed in Lemma 4.2 at the stopping times t_0^i, t_1^i, t_2^i. At t_2^i the position will be cleared and we start the procedure afresh by letting $t_0^{i+1} = t_2^i$. Generally, we assume that the time horizon T is sufficiently large such that the (typically small) levels $s_0^i(1-2c), \ldots, s_0^i(1+2c)$ are reached at least once.

Using the independent increments property of the Black-Scholes model we find that the achieved repeated statistical $\mathscr{G}^i$-arbitrage strategies constitute a statistical $\mathscr{G}$-arbitrage with respect to

$$\mathscr{G} = \sigma\big(S_{t_2^1}, S_{t_2^2}, \ldots, S_{t_2^N}\big).$$

gain p.a.	median	VaR(0.95)	gain/trade	losses	(mean)	∅ N	max. N
33.4	206	5,320	8.74	0.133	-628	3.82	24

Table 1. Simulation results for the *embedded binomial trading strategy* for 1 mio runs. This example serves as benchmark. Gain p.a. denotes the overall average *gain* in the time period of one year, $[0, 1]$; we also show its median and the associated estimated VaR at level 95%. *Gain/trade* denotes the average gain per trade, *losses* denotes the fraction of simulations where the outcome of the trading strategy was negative, and we also show the average of the negative part of the outcomes, titled *mean*. Finally, we also state the average number and maximal number of embedded binomial models.

The constant c and with it the barriers for the hitting times will be chosen in dependence of μ and σ to ensure that we do not lose the statistical arbitrage opportunity. To be more precise we use

$$c = 0.01 \cdot \frac{\mu}{\sigma}$$

which showed a good performance in our simulations. According to Proposition 4.1 there is a statistical arbitrage opportunity if $\frac{P(\omega_2)}{P(\omega_3)} \neq \tilde{q}$. It is easy to check from equation (14) that $\tilde{q} = 1$ in the case considered here.

To guarantee existence of a statistical arbitrage we calculate the path probabilities $P(\omega_2)$, $P(\omega_3)$. The first exit time $\tau = \inf\{t \geq 0 \,|\, S_t \notin (a, b)\}$ from the interval (a, b) satisfies

$$P(S_\tau = a) = \left(\frac{a}{s_0}\right)^\nu \frac{\left(\frac{b}{s_0}\right)^{|\nu|} - \left(\frac{s_0}{b}\right)^{|\nu|}}{\left(\frac{b}{a}\right)^{|\nu|} - \left(\frac{a}{b}\right)^{|\nu|}}, \qquad a < b, \tag{19}$$

where $\nu = \frac{\mu}{\sigma^2} - \frac{1}{2}$, see Borodin and Salminen (2012), formula 3.0.4 in Section 9 of Part II. This in turn yields that

$$\begin{aligned} q = \frac{P(\omega_2)}{P(\omega_3)} &= \frac{P\big(S_{t_1} = s_0(1+c)\big)P\big(S_{t_2} = s_0 | S_{t_1} = s_0(1+c)\big)}{P\big(S_{t_1} = s_0(1-c)\big)P\big(S_{t_2} = s_0 | S_{t_1} = s_0(1-c)\big)} \\ &= \frac{\left(1 - (1-c)^\nu \frac{(1+c)^{|\nu|} - (1+c)^{-|\nu|}}{\left(\frac{1+c}{1-c}\right)^{|\nu|} - \left(\frac{1-c}{1+c}\right)^{|\nu|}}\right)(1+c)^{-\nu} \frac{\left(\frac{1+2c}{1+c}\right)^{|\nu|} - \left(\frac{1+c}{1+2c}\right)^{|\nu|}}{(1+2c)^{|\nu|} - (1+2c)^{-|\nu|}}}{\left((1-c)^\nu \frac{(1+c)^{|\nu|} - (1+c)^{-|\nu|}}{\left(\frac{1+c}{1-c}\right)^{|\nu|} - \left(\frac{1-c}{1+c}\right)^{|\nu|}}\right)\left(1 - \left(\frac{1-2c}{1-c}\right)^\nu \frac{(1-c)^{-|\nu|} - (1-c)^{|\nu|}}{(1-2c)^{-|\nu|} - (1-2c)^{|\nu|}}\right)}. \end{aligned} \tag{20}$$

Clearly, in general $q \neq 1$, such that in these cases statistical arbitrage exists, which we exploit in the following.

From Lemma 4.2 we obtain with $D = 2(q-2)(c\,s_0^i)^3$ that the trading strategy $\boldsymbol{\phi} = (\phi_1, \phi_2^+, \phi_2^-)$ is given by

$$\phi_1 = (2+q)(c\,s_0^i)^2 D^{-1}, \quad \phi_2^+ = (q-4)(c\,s_0^i)^2 D^{-1}, \quad \phi_2^- = -3q(c\,s_0^i)^2 D^{-1}. \tag{21}$$

We call the trading strategy which results by repeated application of ϕ at the respective hitting times the *embedded binomial trading strategy*.

5.2. **Simulation results.** As already mentioned, we simulate a geometric Brownian motion according to Equation (16) with $\mu = 0.1241$, $\sigma = 0.0837$, $S_0 = 2186$, $T = 1$ (year), discretize by 1000 steps and embed the corresponding binomial models repeatedly in this time interval. In this case we have $q = 1.00189$ (rounded to five digits) which is not equal to one and therefore $q \neq \tilde{q}$, i. e. each of the repeated embedded binomial strategies, is a $\mathscr{G}^i$-arbitrage strategy in the associated period i. We denote by N the (random) number of binomial models that are necessary for each simulated diffusion to gain either a profit from trading or to reach T and by G^i the gain or loss of the i-th binomial model. Hence either $\sum_{i=1}^N G^i > 0$ or we record a loss at the final iteration N.

For 1 million runs, we obtain the results presented in Table 1. For each run we record either a gain or a loss from trading. The average gain per simulation run is shown in column one, its median in column two. The distribution of the P&L is skewed to the left with potential large losses with small

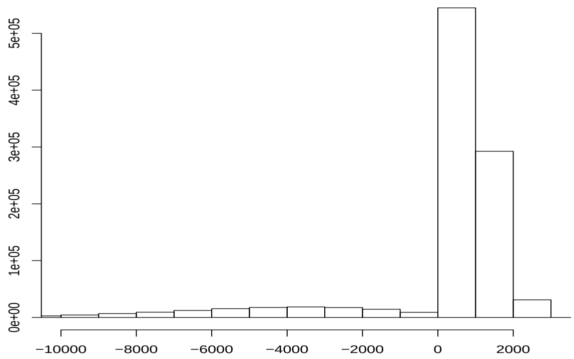

FIGURE 2. Histogram of the profits and losses from the embedded binomial trading strategy used in Table 1 (the y-axis records the frequency, we did 1 million runs).

$c \cdot \sigma/\mu$	gain pa	median	$\mathrm{VaR}_{0.95}$	gain pt	losses	(mean)	$\varnothing\, N$	(max)
0.0025	8,890	48,700	-373	743	0.045	-57,900	12	150
0.005	465	3,810	58,400	66	0.077	-6,210	7	63
0.01	41	206	5,250	11	0.132	-621	4	24
0.02	9	10	371	5	0.185	-50	2	9
0.04	3	2	24	3	0.109	-2	1	4

TABLE 2. Simulations for the embedded binomial trading strategy with varying boundary levels. In the simulations for Table 1 we used $c = 0.01{}^{\mu}/{}_{\sigma}$.

probability which is reflected by a median of 206 in comparison to an average gain of 33. In column 3 we depict the 95% Value-at-Risk which is of size 5,320. Column 4 denotes the average gain per trade which is obtained by dividing the average gain by the average number of trades (i.e. repeated binomial models). In column 5 we show the (fraction of) *losses*, i.e. the fraction of simulated processes exhibiting no gain from trading before reaching the final time T, followed by the mean of the negative part of the outcomes. The average number of trading repeats $\varnothing N$ is followed by the maximal number of trading repeats over all runs (max N).

As becomes clear from Table 1 we can record an overall profit for most cases. We have a negative outcome in 13.3 percent in average of all simulations with an average size of -628. The median of the profits is about 200, with a smaller average of about 30. The risk measured by the Value-at-Risk at 95% is 5,320 pointing to the fact that the average gain by the statistical arbitrage is (of course) not without risk. For clarification, we plot the associated histogram of the P&L in Figure 2.

Although the actual amount of the profit depends on many parameters we can confirm the possibility of statistical arbitrage. Besides, we see that on average our multi-period binomial model has a small number of periods and the number of periods does not explode, which is important with a view on trading costs.

Varying barrier levels. The most interesting parameter turns out to be the parameter c. It decodes the varying the barrier level and the results are given in Table 2. It turns out that this parameter allows to balance gains and risk very well.

First, the smaller the parameter c is chosen, the higher are the gains in general. The additional gain does imply an increase of risk: most prominently, the mean of the negative part of the outcome decreases with c. The Value-at-Risk confirms the increase of risk with decreasing c, except for the lowest $c = 0.0025$. In this case, the probability of having large losses is below 5%, such that the Value-at-Risk at level 0.95% does no longer see this risk (while it is of course still present).

A high value of c corresponds intuitively to a larger step sizes, which leads to less trades on average. The largest value of c gives a statistical arbitrage with small gain and smallest risk.

η	gain pa	median	$\mathrm{VaR}_{0.95}$	gain pt	losses	(mean)	$\varnothing$ N	(max)
0.50	170	4,360	94,500	36	0.13	-11,000	5	30
0.75	109	1,730	38,100	23	0.13	-4,400	5	30
1.00	64	913	20,400	14	0.12	-2,340	5	30
1.25	77	561	12,400	17	0.12	-1,400	5	30
2.00	42	197	4,430	9	0.11	-490	4	31

TABLE 3. Simulations for the embedded binomial trading strategy with different values of the drift μ (and hence η), fixed $\sigma = 0.1$ and $n = 250,000$ runs; gain p.a. denotes the gain per year, gain p.t. denotes gain per trade.

η	gain pa	median	$\mathrm{VaR}_{0.95}$	gain pt	losses	(mean)	$\varnothing$ N	(max)
0.50	74,500	222,000	-48,400	4,340	0.036	-2,770,000	17	270
0.75	6,020	59,900	480,000	582	0.056	-79,400	10	120
1.00	241	4,710	80,500	37	0.090	-8,520	7	51
1.25	67	541	12,700	16	0.124	-1,460	4	28
2.00	8	6	165	5	0.144	-22	2	9

TABLE 4. Simulations for the embedded binomial trading strategy with different values of the volatility (and hence η), fixed $\mu = 0.1$; gain pa denotes the gain per year, gain pt denotes gain per trade.

The role of drift and volatility. For the investor it is of interest which drift and which volatility of an asset promises a good profit. To investigate this question we define the fraction

$$\eta := \frac{\mu}{\sigma}$$

and show simulation results for different values of η. In Table 3 we fix the volatility σ and consider varying drift, while in Table 4 we fix the drift μ and consider varying volatility.

Larger values of η point to a high drift relative to volatility situations which we would expect to be very well exploitable. In fact, our simulations show quite the contrary: we observe large gains when η is actually small, while for larger η we observe only minor gains. More precisely, for fixed σ we obtain decreasing gains for increasing drift, while for fixed μ we observe increasing gains for increasing volatility. This effect is much more pronounced for the latter case (increasing σ). Already from the results with varying step sizes in Table 2 such an effect was to be expected, as higher values of η lead to larger step sizes here and to lower gains. Intuitively, larger volatility implies more repetitions and therefore a higher likelihood for the statistical arbitrage to end up with gains. This is also reflected by increasing values of N in Table 4.

5.3. **Follow-the-trend strategy.** As we have seen in the previous section, embedding a binomial model into continuous time is not able to exploit a large drift. This motivates the introduction of a further step into the embedded model in order to *exploit existing trends* in the underlying. We focus on an *upward trend*, while the strategy is easily adopted to the case for a downward trend. We consider two-step binomial embedding: first, we specify barriers (up/down) as previously. If we twice observed up movements, we expect an upward trend and exploit this in a further step. Consequently, here we will consider four stopping times (for iteration i): initial time τ_0^i, and stopping times τ_1^i, τ_2^i as previously and, in addition τ_3^i. Most notably, this modelling implies a different choice of the filtration $\mathscr{G}$, see Equation (25).

The associated strategy is to trade in the following way: the first trading occurs as previously at the first time when the barriers $s(1+c)$ or $s(1-c)$ are hit. The next trading takes place when the neighbouring barriers are hit, in the first case s or $s(1+2c)$ and in the second case s or $s(1-2c)$, respectively. If a trend was detected (i.e. the upper barrier $s(1+2c)$ was hit, as we consider the case of a positive drift), trading continues until a suitable stopping time.

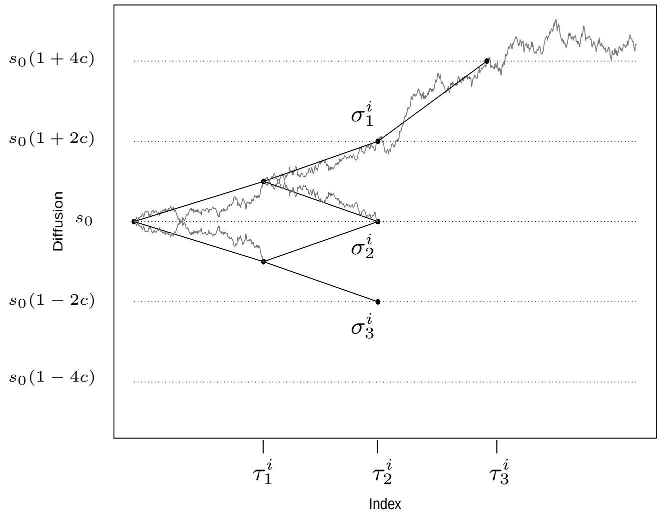

FIGURE 3. Illustration of the stopping times defined in (22), (23) resp. (24). The first stopping takes place when the process reaches either the first upper or lower boundary $s_0^i(1 \pm c)$. Starting from the upper boundary the next stopping takes place if the process increases to the level $s_0^i(1+2c)$, decreases to the level $s_0^i(1-2c)$ or crosses the level s_0. In case the process reached the upper level a third stopping occurs at τ_3^i.

More formally, this leads to the following procedure: let i denote the current step of our iteration. We initialize at time $\tau_0^0 = 0$. Otherwise consider the initial time of our next iteration given by the the time where we finished the last repetition and denote this time by τ_0^i and the according level by $s_0^i = S_{\tau_0^i}$. Then, using again the property that S is continuous, we define the following successive stopping times: first, analogously to t_1^i from Equation (17), let

$$\tau_1^i = \inf\left\{t \in (\tau_0^i, T] \,|\, S_t \geq s_0^i(1+c) \text{ or } S_t \leq s_0^i(1-c)\right\}. \tag{22}$$

In the same manner the second stopping occurs if either the upper level is reached, or the mid-level is crossed, or the bottom level is reached. The levels of course differ depending on whether $S_{\tau_1^i} = s_0^i(1+c)$ or $S_{\tau_1^i} = s_0^i(1-c)$. In this regard, we define (for the first case)

$$\sigma_1^i = \inf\left\{t \in (\tau_1^i, T] \,|\, S_t \geq s_0^i(1+2c)\right\}, \qquad \sigma_2^i = \inf\left\{t \in (\tau_1^i, T] \,|\, S_t \leq s_0^i\right\}.$$

For the second case, we set

$$\sigma_3^i = \inf\left\{t \in (\tau_1^i, T] \,|\, S_t \leq s_0^i(1-2c)\right\}, \qquad \sigma_4^i = \inf\left\{t \in (\tau_1^i, T] \,|\, S_t \geq s_0^i\right\}.$$

Altogether we obtain that

$$\tau_2^i = \begin{cases} \sigma_1^i \wedge \sigma_2^i & \text{if } S_{\tau_1^i} = s_0^i(1+c), \\ \sigma_3^i \wedge \sigma_4^i & \text{otherwise.} \end{cases} \tag{23}$$

Finally, we set

$$\tau_3^i = \begin{cases} \inf\left\{t \in (\tau_2^i, T] \,|\, S_t \leq s_0 \text{ or } S_t \geq s_0^i(1+4c)\right\}, & \text{if } S_{\tau_2^i} = s_0^i(1+2c), \\ \tau_2^i, & \text{otherwise.} \end{cases} \tag{24}$$

Denote by $\tau^{\max}$ the last stopping time of $\tau_3^1, \tau_3^2, \ldots$ which lies before T. Then the statistical arbitrages traded on the partition of $S_{\tau^{\max}}$ generated by the values $s_0(1+2kc)$, $k = 0, 1, 2, \ldots$ which defines the $\mathscr{G}$ on the path space of the diffusion.

Trading will be executed at times τ_1^i to τ_3^i when the process reaches one of the predefined boundaries (or trading time is over). At time τ_2^i we check if a positive trend persists and trade on this trend. Recall the trading strategy $\boldsymbol{\phi} = (\phi_1, \phi_2^+, \phi_2^-)$ from equation (21). First, trading at the first two times

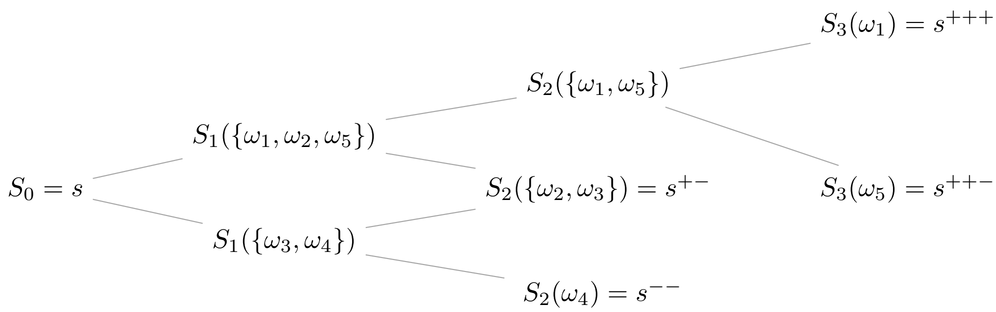

FIGURE 4. The embedded binomial model for the follow-the-trend strategy with positive drift. The filtration generated by the final states is generated by each $\{\omega_i\}$ for $i = 1, 4, 5$ and $\{\omega_2, \omega_3\}$. We also denote the resulting outcomes by $s = s_0$, s^+, s^-, ... and indicate this notation at some places.

is executed as previously at times t_0^i, t_1^i, see Lemma 4.2: we hold on $[\tau_0^i, \tau_1^i)$ the fraction ϕ_1 shares of S. After reaching $s_0^i(1+c)$ ($s_0^i(1-c)$, respectively) at time τ_1^i the trading strategy changes to holding ϕ_2^+ (ϕ_2^-) shares of S until τ_2^i. The next trading can be split into the following three cases:

(i) $\tau_2^i = \sigma_1^i$: in this case we reached the upper level $s_0^i(1+2c)$ and follow the (upward) trend by holding ϕ_3^{++} shares of S. This position will be equalized at τ_3^i or if the final time is reached.
(ii) τ_2^i equals σ_2^i or σ_4^i: from the state $s_0^i(1+c)$ resp. $s_0^i(1-c)$ we arrived back at s_0^i (or below resp. above). No trend was detected and the embedded binomial trading strategy ends by liquidating the position.
(iii) τ_2^i equals σ_4^i: again, no (upward) trend was detected and the strategy ends by liquidation the position.

Since Lemma 4.2 treats a related, but slightly different case we explicitly check in the following that the embedded binomial model indeed allows for statistical arbitrage.

The embedded binomial follow-the-trend strategy. We consider $\tilde{\Omega} = \{\omega_1, \dots, \omega_5\}$ as depicted in Figure 4. Let $S_0 = s_0 \in \mathbb{R}_{\geq 0}$ and S_1 take the two values s^+ and s^- such that

$$S_1(\omega_1) = S_1(\omega_2) = S_1(\omega_5) = s^+, \qquad S_1(\omega_3) = S_1(\omega_4) = s^-.$$

At time 2 we have the three possibilities $S_2(\omega_1) = S_2(\omega_5) = s^{++}$, $S_2(\omega_2) = S_2(\omega_3) = s^{+-}$ and $S_2(\omega_4) = s^{--}$. In the cases of $\omega_2, \dots, \omega_4$ the model stops. If, however, we saw two up-movements, the model continues and ends up at time 3 in the states $S_3(\omega_1) = s^{+++}$ or $S_3(\omega_5) = s^{++-}$. We assume without loss of generality that $s^+ > s_0$, $s^- < s_0$, and $s^{++} > s^+$, $s^- < s^{+-} < s^+$, and $s^{--} < s^-$ as well as $s^{++-} < s^{++} < s^{+++}$, i. e. we consider binomial models as presented in Figure 4.

The dynamic trading strategies can be described by

$$V_3(\phi) = \phi_1 \Delta S_1 + \phi_2 \Delta S_2 + \phi_3 \Delta S_3,$$

with ϕ_1, ϕ_2^+, ϕ_2^- and ϕ_3^{++} being the respective values in the states $\tilde{\Omega}$, $\{\omega_1, \omega_2, \omega_5\}$, $\{\omega_3, \omega_4\}$ and $\{\omega_1, \omega_5\}$ at times 1, 2, and 3, respectively. Moreover, we choose

$$\tilde{\mathscr{G}} = \sigma(\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}, \{\omega_5\}), \tag{25}$$

i.e. the σ-field generated by the final states of the embedded binomial model. The following lemma shows that there is statistical arbitrage in the follow-the-trend strategy if there is statistical arbitrage in the recombining two-period sub-model consisting only of the first two periods.

Denote

$$\gamma = \frac{1}{D} \begin{pmatrix} q \Delta S_2(\omega_2) \Delta S_2(\omega_4) \\ \Delta S_1(\omega_4) \Delta S_2(\omega_3) - \big(q \Delta S_1(\omega_2) + \Delta S_1(\omega_3)\big) \Delta S_2(\omega_4) \\ -q \Delta S_2(\omega_2) \Delta S_1(\omega_4) \end{pmatrix} \tag{26}$$

with D given in Lemma 4.2. We recall from Proposition 4.1, that statistical arbitrages exist if and only if $P(\omega_2)/P(\omega_3) \neq \tilde{q}$. The following results shows, that in the follow-the-trend model there is statistical arbitrage under the same condition.

gain pa	mean	$\mathrm{VaR}_{0.95}$	gain pt	losses	(mean)	$\varnothing\, N$	(max)
27.8	164	4,180	9.17	0.171	-554	3	21

TABLE 5. Simulations for the follow-the-trend strategy for 1 mio runs. In comparison to Table 1 (where the notation is explained) we find slightly smaller gains together with a smaller risk.

c	gain pa	median	$\mathrm{VaR}_{0.95}$	gain pt	losses	(mean)	$\varnothing\, N$	(max)
$0.005\,\mu/\sigma$	404	3,300	51,300	71.1	0.098	-5,590	6	44
$0.01\,\mu/\sigma$	32	162	4,130	10.7	0.169	-548	3	18
$0.02\,\mu/\sigma$	6	8	272	3.9	0.238	-45	2	7
$0.04\,\mu/\sigma$	3	1	23	2.6	0.122	-2	1	3

TABLE 6. Simulations for the follow-the-trend strategy with varying barrier levels c. In the simulations for Table 5 we used $c = 0.01\,\mu/\sigma$.

Proposition 5.1. *If $\boldsymbol{\phi}$ is the strategy from Lemma 4.2, then for any $\alpha \geq 0$, $\boldsymbol{\psi} = (\psi_1, \psi_2^+, \psi_2^-, \psi_3^{++})$ with*

$$\psi_3^{++} = \frac{1-\alpha}{\Delta S_3(\omega_1) - \Delta S_3(\omega_5)}$$

and

$$\begin{pmatrix} \psi_1 \\ \psi_2^+ \\ \psi_2^- \end{pmatrix} = \boldsymbol{\phi} - \Delta S_3(\omega_1)\psi_3^{++}\boldsymbol{\gamma}$$

is a $\tilde{\mathscr{G}}$-arbitrage strategy, if $\frac{P(\omega_2)}{P(\omega_3)} \neq \tilde{q}$.

The proof is deferred to the Appendix. Note that the possible choice $\alpha = 1$ leads to $\psi_3^{++} = 0$, such that in this case the statistical arbitrage in the first two periods is exploited and the strategy coincides with that of Lemma 4.2.

Simulation results. We study the performance of the follow-the-trend strategy on the basis of various simulations and compare it to the results of the embedded binomial strategies. As previously, we simulate a geometric Brownian motion according to Equation (16) with $\mu = 0.1241$, $\sigma = 0.0837$, $S_0 = 2186$, $T = 1$ (year), discretize by 1000 steps and embed the according models repeatedly in this time interval. In this case, Proposition 5.1 grants the existence of statistical arbitrage which we will exploit in the following.

Contrary to the intention of improving the average gain of the follow-the-trend strategy, the simulations show that this goal is not achieved. But, in general, the follow-the-trend strategy leads to a reduction of risk compared to the embedded-binomial trading strategy, visible through the reduced Value-at-Risk in Tables 5 to 8. The reduction of the average gain and its mean can be explained from the observations in Section 4.3: the follow-the-trend-strategy introduces additional scenarios with smaller gains (compare Figure 4). This leads to a reduction of the average gain and, at the same time, to a reduction of risk.

The results from Table 6 to 8 show a similar dependence on the choice of the parameters and of the barrier of the follow-the-trend strategy compared to the embedded binomial strategy. In general, we record smaller gains together with smaller risk with one exception: the last line of Table 8 shows that a small η (and hence a large σ) allows the follow-the-trend strategy to exploit the existing (although small) positive trend in the data better. Of course, this comes with a higher risk, which is clearly visible.

Summarizing, the follow-the-trend strategy shows (in general) smaller gains together with a smaller risk. The follow-the-trend strategy is, however, able to exploit a positive trend when σ is very small.

η	gain pa	median	$\mathrm{VaR}_{0.95}$	gain pt	losses	(mean)	$\varnothing$ N	(max)
0.50	122	3,500	76,200	31	0.16	-9,780	4	24
0.75	99	1,390	30,400	26	0.16	-3,890	4	22
1.00	78	734	16,200	20	0.15	-2,050	4	23
1.25	54	452	9,950	15	0.15	-1,260	4	23
2.00	34	162	3,570	10	0.14	-436	3	21

TABLE 7. Simulations for the follow-the-trend strategy with varying values of the drift (and hence $\eta = \mu/\sigma$) with fixed $\sigma = 0.1$.

η	gain pa	median	$\mathrm{VaR}_{0.95}$	gain pt	losses	(mean)	$\varnothing$ N	(max)
0.50	2,010	40,700	586,000	284	0.09	-62,500	7	58
0.75	292	3,930	69,200	60	0.12	-7,940	5	34
1.00	44	732	16,400	11	0.15	-2,080	4	24
1.25	27	200	5,330	9	0.18	-729	3	17
2.00	10	15	469	5	0.20	-68	2	9

TABLE 8. Simulations for the follow-the-trend strategy with varying values of the volatility σ and fixed $\mu = 0.1$.

5.4. **Partition strategies on the final value.** In this section we study statistical arbitrage with respect to $\mathscr{G}^{\text{fin}}$ generated by

$$\{S_T \geq s_0\} = \{\omega_1, \omega_2, \omega_3\}, \qquad \text{and } \{S_T < s_0\} = \{\omega_4, \omega_5\}. \tag{27}$$

This σ-field corresponds to the two scenarios that the value of the asset increased or decreased at time T. The statistical $\mathscr{G}^{\text{fin}}$-arbitrage corresponds to a strategy which yields an average profit in both of these scenarios.

As an example, we continue in the setting of the follow-the-trend model considered in the previous Section 5.3, although other settings are clearly possible. Recall that this means we are focusing on an upward trend. We add the assumption that $s^{++-} < s_0$ such that also the third period allows for interesting outcomes (below *or* above s_0, compare Figure 4). The new σ-field will lead to a different trading strategy as we detail in the following.

Proposition 5.2. *In the follow-the-trend model with $s^{++-} < s_0$ there is $\mathscr{G}^{\text{fin}}$-arbitrage if*

$$\begin{aligned}\Big(\psi_1 \Delta S_1(\omega_1) + \psi_2^+ \Delta S_2(\omega_1) + \psi_3^{++} \Delta S_3(\omega_1)\Big) + \Big(\psi_1 \Delta S_1(\omega_3) + \psi_2^- \Delta S_2(\omega_3)\Big) \frac{P(\omega_3)}{P(\omega_1)} \\ + \Big(\psi_1 \Delta S_1(\omega_2) + \psi_2^+ \Delta S_2(\omega_2)\Big) \frac{P(\omega_2)}{P(\omega_1)} \geq 0,\end{aligned} \tag{28}$$

$$\Big(\psi_1 \Delta S_1(\omega_4) + \psi_2^- \Delta S_2(\omega_4)\Big) + \Big(\psi_1 \Delta S_1(\omega_5) + \psi_2^+ \Delta S_2(\omega_5) + \psi_3^{++} \Delta S_3(\omega_5)\Big) \frac{P(\omega_5)}{P(\omega_4)} \geq 0 \tag{29}$$

and, in addition, at least one of the inequalities is strict.

The proof is immediate. Note that there is a lot of freedom in choosing such strategies. Indeed, we will pursue choosing a strategy matching our previous strategies for better comparability.

Example 5.3. We consider a special case of (28), (29): we additionally assume that the first line of Equation (28) and the first line of Equation (29) is non-negative. Then, the strategy $\boldsymbol{\psi}$ is a

$\mathscr{G}^{\text{fin}}$-arbitrage if

$$\begin{aligned} \psi_1 \Delta S_1(\omega_1) + \psi_2^+ \Delta S_2(\omega_1) + \psi_3^{++} \Delta S_3(\omega_1) &\geq 0, \\ \psi_1 \Delta S_1(\omega_3) + \psi_2^- \Delta S_2(\omega_3) + \Big(\psi_1 \Delta S_1(\omega_1) + \psi_2^+ \Delta S_2(\omega_2)\Big) \frac{P(\omega_2)}{P(\omega_3)} &\geq 0, \\ \psi_1 \Delta S_1(\omega_3) + \psi_2^- \Delta S_2(\omega_4) &\geq 0 \\ \psi_1 \Delta S_1(\omega_1) + \psi_2^+ \Delta S_2(\omega_1) + \psi_3^{++} \Delta S_3(\omega_5) &\geq 0, \end{aligned} \tag{30}$$

and at least one inequality is strict. Note that we used $\Delta S_1(\omega_3) = \Delta S_1(\omega_4)$, $\Delta S_1(\omega_1) = \Delta S_1(\omega_2) = \Delta S_1(\omega_5)$ and $\Delta S_2(\omega_1) = \Delta S_2(\omega_5)$ from Section 5.3. This choice is similar to the previously studied partition strategies and we compute a strategy explicitly. In this regard, define the matrix A by

$$A = \begin{pmatrix} \Delta S_1(\omega_1) & \Delta S_2(\omega_1) & 0 & \Delta S_3(\omega_1) \\ \Delta S_1(\omega_3) + r\Delta S_1(\omega_1) & r\Delta S_2(\omega_2) & \Delta S_2(\omega_3) & 0 \\ \Delta S_1(\omega_3) & 0 & \Delta S_2(\omega_4) & 0 \\ \Delta S_1(\omega_1) & \Delta S_2(\omega_1) & 0 & \Delta S_3(\omega_5) \end{pmatrix}$$

with $r = \frac{P(\omega_2)}{P(\omega_3)}$. If A is invertible, for any $\alpha \geq 0$, the strategy $\boldsymbol{\psi}$ given by

$$\psi_3^{++} = \frac{1-\alpha}{\Delta S_3(\omega_1) - \Delta S_3(\omega_5)}$$

and

$$\begin{pmatrix} \psi_1 \\ \psi_2^+ \\ \psi_2^- \end{pmatrix} = \boldsymbol{\phi} - \Delta S_3(\omega_1)\psi_3^{++}\boldsymbol{\gamma}$$

is a $\mathscr{G}^{\text{fin}}$-arbitrage. Here, $\boldsymbol{\phi} = \frac{1}{D}(\xi^1, \xi^2, \xi^3)$ with

$$\begin{aligned} \xi^1 &= \Big(r\Delta S_2(\omega_2) - \Delta S_2(\omega_1)\Big)\Delta S_2(\omega_4) + \Delta S_2(\omega_1)\Delta S_2(\omega_3), \\ \xi^2 &= \Big(\Delta S_1(\omega_3) - \Delta S_1(\omega_1)\Big)\Delta S_2(\omega_3) + \Big(\Delta S_1(\omega_1) - \Delta S_1(\omega_3) - r\Delta S_1(\omega_1)\Big)\Delta S_2(\omega_4), \\ \xi^3 &= r\Delta S_1(\omega_1)\Big(\Delta S_2(\omega_2) - \Delta S_2(\omega_1)\Big) - r\Delta S_2(\omega_2)\Delta S_1(\omega_3), \end{aligned}$$

and

$$D = \Big(r\Delta S_1(\omega_1)\Delta S_2(\omega_2) - (\Delta S_1(\omega_3) + r\Delta S_1(\omega_2))\Delta S_2(\omega_1)\Big)\Delta S_2(\omega_4) + \Delta S_1(\omega_3)\Delta S_2(\omega_1)\Delta S_2(\omega_3),$$

computed analogously to Lemma 4.2. In addition,

$$\boldsymbol{\gamma} = \frac{1}{D}\begin{pmatrix} r\Delta S_2(\omega_2)\Delta S_2(\omega_4) \\ \Delta S_1(\omega_3)\Delta S_2(\omega_3) - (r\Delta S_1(\omega_1) + \Delta S_1(\omega_3))\Delta S_2(\omega_4) \\ -r\Delta S_2(\omega_2)\Delta S_1(\omega_3) \end{pmatrix},$$

and the computation of the strategy is finished. ◇

Simulation results. Again, we study the performance of the strategy, this time the strategy derived in Example 5.3 with a partition (above/below) on the final value of the stock. We perform various simulations. As previously, we simulate a geometric Brownian motion according to Equation (16) with $\mu = 0.1241$, $\sigma = 0.0837$, $S_0 = 2186$, $T = 1$ (year), discretize by 1000 steps and embed the according models repeatedly in this time interval. The properties for existence of a statistical arbitrage in this setting are confirmed numerically.

As pointed out before, the statistical arbitrages are with respect to different σ-algebras. By our variant of $\mathscr{G}^{\text{fin}}$-arbitrage chosen in Example 5.3 we find very similar results to the follow-the-trend strategy as one can see in Tables 9 to 12.

gain pa	median	$\mathrm{VaR}_{0.95}$	gain pt	losses	(mean)	$\varnothing N$	(max)
28.6	167	4,290	8.76	0.158	-544	3	20

TABLE 9. Statistical $\mathscr{G}^{\text{fin}}$-arbitrage trading strategy simulation results for 1 mio simulations with $c^i = 0.01\,\eta\,S_{\sigma_0^i}$.

c	gain pa	median	$\mathrm{VaR}_{0.95}$	gain pt	losses	(mean)	$\varnothing N$	(max)
$0.005\,\mu/\sigma$	356	3,280	51,500	58	0.09	-5,510	6	49
$0.01\,\mu/\sigma$	28	166	4,290	9	0.15	-543	3	19
$0.02\,\mu/\sigma$	6	8	288	4	0.22	-44	2	8
$0.04\,\mu/\sigma$	3	1	22	3	0.12	-2	1	4

TABLE 10. Simulation results for the statistical $\mathscr{G}^{\text{fin}}$-arbitrage trading strategy with varying boundaries of the embedded binomial model.

η	gain pa	median	$\mathrm{VaR}_{0.95}$	gain pt	losses	(mean)	$\varnothing N$	(max)
0.50	112	3,560	77,700	26.7	0.15	-9,600	4	25
0.75	97	1,430	31,200	23.7	0.14	-3,830	4	26
1.00	73	751	16,600	18.3	0.14	-2,020	4	26
1.25	55	458	10,100	13.9	0.14	-1,230	4	24
2.00	34	163	3,600	9.15	0.13	-428	3	25

TABLE 11. Statistical $\mathscr{G}^{\text{fin}}$-arbitrage trading strategy for varying μ but with fixed $\sigma = 0.01$.

η	gain pa	median	$\mathrm{VaR}_{0.95}$	gain pt	losses	(losses)	$\varnothing N$	(max)
0.75	203	3,890	69,800	38	0.11	-7,810	5	37
1.00	71	752	16,600	18	0.14	-2,020	4	25
1.25	28	205	5,500	9	0.17	-715	3	18
2.00	10	15	494	5	0.19	-67	2	11

TABLE 12. Statistical $\mathscr{G}^{\text{fin}}$-arbitrage trading strategy for varying σ but fixed $\mu = 0.1$.

5.5. **Summary on the different embedded strategies.** The previous results confirm that all three introduced strategies are statistical $\mathscr{G}$-arbitrage strategies with respect to the corresponding choices of $\mathscr{G}$. Although we observe similar patterns through all strategies, like higher gains for smaller boundaries or an average profit decreasing in η, there are significant differences between the strategies:

(i) the *average profit* achieved is best for the embedded binomial strategy.
(ii) The follow-the-trend strategy and the $\mathscr{G}^{\text{fin}}$-arbitrage strategy show similar behaviour: while showing smaller gains on average, these two strategies have smaller risk.

5.6. **The Bachelier model in discrete time.** The explicit cacluations in the previous sections were given for the complete Black-Scholes model. Here, we provide a simple example to illustrate that statistical arbitrage can be exploited also in incomplete models. To this end, we consider a (modified) Bachelier model in discrete time,

$$S_t = \sum_{i=1}^{t} \xi_i, \qquad t = 0, 1, 2.$$

We assume that ξ_1 and ξ_2 are independent and normally distributed with standard variance. We choose $\mathscr{G} = \sigma(S_2)$.

If $E[\xi_1] = E[\xi_2] = 0$, then the measure P is already a martingale measure. Hence the density $dQ/dP = 1$ and therefore $\mathscr{G}$-measurable. Proposition 2.4 yields that there is no statistical $\mathscr{G}$-arbitrage. We hence assume $E[\xi_1] = 0$ and $E[\xi_2] = \mu_2 \neq 0$. In this case, the conditional distribution of ξ_1 given S_2 is again normal and computes to

$$\xi_1 \ |S_2 = \mathcal{N}(\rho(S_2 - \mu_2), 1 - \rho^2),$$

where the correlation is $\rho = \frac{E[\xi_1(\xi_1+\xi_2)]}{\sqrt{1}\sqrt{2}} = \frac{1}{\sqrt{2}}$. The self-financing trading strategy ϕ is defined by the deterministic position ϕ_1 at time 0 and the ξ_1-measurable position $\phi_2 = \phi_2(\xi_1)$ at time 1. Statistical arbitrage is equivalent to

$$\phi_1 E[\xi_1|S_2] + E\big[\phi_2(\xi_1) \cdot (S_2 - \xi_1)|S_2\big] \geq 0, \tag{31}$$

and being not equal to zero almost surely. Of course, there are many possible choices for ϕ_1 and ϕ_2 and we provide a simple example: choose ϕ_2 of the form $\phi_2(x) = ax\mathbb{1}_{x>\alpha} + bx\mathbb{1}_{x<\beta}$. In this case the left hand side of (31) can be computed and we obtain that[1]

$$\begin{aligned}
&\phi_1 E[\xi_1|S_2] + E\big[\phi_2(\xi_1) \cdot (S_2 - \xi_1)|S_2\big] \\
&\quad = \phi_1\rho\mu + aS_2\Big[\mu(1-\Phi(c_\alpha)) + \frac{\sigma}{\sqrt{2\pi}}e^{-\frac{1}{2}c_\alpha^2}\Big] - a\Big[(\mu^2+\sigma^2)(1-\Phi(c_\alpha)) + \frac{1}{\sqrt{2\pi}}(2\mu\sigma + \sigma^2 c_\alpha)e^{-\frac{1}{2}c_\alpha^2}\Big] \\
&\quad + bS_2\Big[\mu\Phi(c_\beta) - \frac{\sigma}{\sqrt{2\pi}}e^{-\frac{1}{2}c_\beta^2}\Big] - b\Big[(\mu^2+\sigma^2)\Phi(c_\beta) - \frac{1}{\sqrt{2\pi}}(2\mu\sigma + \sigma^2 c_\beta)e^{-\frac{1}{2}c_\beta^2}\Big];
\end{aligned}$$

where we use $\mu = \mu(S_2) := \rho(S_2 - \mu)$, $\sigma = 2^{-1/2}$, $c_\alpha = c_\alpha(S_2) := {}^{\alpha-\mu(S_2)}/_{\sigma}$, and $c_\beta = c_\beta(S_2) := {}^{\beta-\mu(S_2)}/_{\sigma}$. For the parameters $\mu_2 = 10$, $\phi_1 = 1$, $a = 1$, $\alpha = 0$, $b = -2$, $\beta = -1$ we numerically verify that this is indeed a statistical arbitrage.

6. Application to market data

In this section we apply the previously studied approaches to real stock data. A full application study of statistical arbitrages arising from our previous considerations is beyond the scope of this article. However, we show a small number of applications to point out certain subtleties and difficulties with the applications, leaving a thorough study for future work.

Three major indices. We consider the S&P 500, the Dow Jones Industrial and the FTSE index from Jan 2000 to Nov 2019. Even if the prices will not follow exactly a Black-Scholes model, we see in this section that the introduced strategies are able to ensure statistical arbitrage. Trading strategies are used by implementing the embedded binomial strategy from Section 5.1 while the parameters (μ, σ) of the geometric Brownian motion have to be estimated. The estimated parameters have two effects: on the one side, they are necessary to compute the portfolio weights, and, on the other side, they are used to choose the grid size in the embedding appropriately. Our previous choice choosing the grid proportional to $\mu\sigma^{-1}$ will naturally depend on the properties of the estimators. Since the drift μ is difficult to estimate, we expect difficulties here. Indeed, it turns out that a drift estimated to be close to zero will imply a very small grid size. Our considerations in Section 5.2 showed that a small grid size implies a large risk, which might be undesired.

As a first solution to this we simply fix the drift to $\mu = 0.1$ and estimate the volatility with a sliding-window approach with a window length of 3 years. Already these results are quite promising, compare Table 13. Compared to the simulation studies, monotonicity with respect to the increasing parameter c does no longer hold: this is because now we are working on a single sample path and not averaging over many paths. Increasing c leads to a completely new construction of the grid, with possibly disruptive outcome. In general the picture from our previous findings can be confirmed on data (results not shown): first, increasing c lowers the risk of the strategy. In particular for small c this implies a high volatility of the outcome (since we have moderate number of trades, ranging from 15 to 70 for $c = 0.1$ or $c = 0.2$). For larger c the strategies get less risky and the overall picture improves.

However, the performance on the FTSE index could be improved. In a second attempt, we instead estimate μ also with a sliding window of the same length. This improves the result dramatically, but

[1]The computation is delegated to the appendix.

c	S&P 500	DJI	FTSE	FTSE (min c)
0.1	-7791	-390	4618	4300
0.2	-574	1.54	-1171	4300
0.3	259	3.35	0.526	4300
0.4	0.186	16.3	-115	4424
0.5	66.7	7.68	1.12	4300
0.6	41.7	8.08	7.39	7647

TABLE 13. *Gains per traded assets (GPTA)* for the embedded binomial strategy, applied to the three indices. Volatility was estimated by maximum-likelihood methods with a rolling window of length 3 years and the drift set equal to $\mu = 0.1$. The boundaries for the embedding were chosen as $S_{\sigma_i} \cdot c \cdot {}^{\mu}/{\sigma}$. The number of trades range from 2 to 70, which is moderate. The far right column shows the strategy when μ is estimated, but $c\mu\sigma^{-1}$ is capped at a minimum level of 0.1. This strategy clearly takes up the positive impression from the FTSE strategy with $c = 0.1$.

this also induces a high risk of the strategy, which might be undesirable. To balance these two aspects we introduced a lower bound on $c\hat{\mu}(\hat{\sigma})^{-1}$, set to 0.1. The outcomes are shown on the far right column in Table 13 underlining the improvement with this strategy.

Summarizing, the results of our application to market data shows that the right choice of the paramter c is very important. For each studied index, we could achieve a positive outcome with the right choice of c. However, our restricted application study indicates the requirement of further detailed empirical study of the strategies suggested in this paper, and in particular on the optimal choice of boundaries for the embedding. From the many open problems of interest we mention the following three:

(i) Are the proposed strategies in this paper competitive to pair trading or related strategies in mean-reverting models?
(ii) Can one improve the embedded binomial strategy by considering more general tree-like embedding models (for example trinomial or more sophisticated ones)?
(iii) Is the method of capping the boundary level c a generally good compromise between risk and profit ? What is the best capping level?

7. CONCLUSION

We introduce the concept NSA($\mathscr{G}$) of no statistical $\mathscr{G}$-arbitrage w.r.t. trading strategies, give a sufficient condition for its absence and show its equivalence to non-existence of generalized statistical $\mathscr{G}$-arbitrage strategies, $\overline{\text{NSA}}(\mathscr{G})$. Moreover, we examine various profitable strategies both on simulated and on market data. The choice of the σ-algebra $\mathscr{G}$ is either motivated by the aim to generate profitable strategies in average over certain pre-determined scenarios or, alternatively, it can be used as a technical tool to generate profitable strategies.

Our data experiments based on simulated data give hints on the choice of parameters for various algorithms in order to gain a good balance of the algorithms between profit and risk. We also show the potential usefulness of the algorithms on simple portfolios of market data. For the application in practice the strategies introduced have to be investigated in more detail, adapted to large portfolios, and adjusted to include transaction costs. This is subject to future research.

APPENDIX A. PROOFS

Proof of Lemma 2.6. Note that equations (7) reads $A\xi \geq 0$ with

$$A = \begin{pmatrix} \Delta S_1(\omega_2) & \Delta S_2(\omega_2) & 0 \\ \Delta S_1(\omega_6) & 0 & \Delta S_2(\omega_6) \\ \Delta S_1(\omega_1)\nu_1 + \Delta S_1(\omega_4) & \Delta S_2(\omega_1)\nu_1 & \Delta S_2(\omega_4) \\ \Delta S_1(\omega_3)\nu_2 + \Delta S_1(\omega_5) & \Delta S_2(\omega_3)\nu_2 & \Delta S_2(\omega_5) \end{pmatrix}.$$

We do a change of basis for the mapping A and substitute the vector in the first column. This leads to a matrix $\tilde{A}$,

$$\tilde{A} = \begin{pmatrix} 0 & \Delta S_2(\omega_2) & 0 \\ 0 & 0 & \Delta S_2(\omega_6) \\ B_1 & \Delta S_2(\omega_1)\nu_1 & \Delta S_2(\omega_4) \\ B_2 & \Delta S_2(\omega_3)\nu_2 & \Delta S_2(\omega_5) \end{pmatrix}$$

where

$$B_1 = \nu_1 \left(\Delta S_1(\omega_1) - \Delta S_2(\omega_1) \frac{\Delta S_1(\omega_2)}{\Delta S_2(\omega_2)} \right) + \Delta S_1(\omega_4) - \Delta S_2(\omega_4) \frac{\Delta S_1(\omega_6)}{\Delta S_2(\omega_6)}$$
$$B_2 = \nu_2 \left(\Delta S_1(\omega_3) - \Delta S_2(\omega_3) \frac{\Delta S_1(\omega_2)}{\Delta S_2(\omega_2)} \right) + \Delta S_1(\omega_5) - \Delta S_2(\omega_5) \frac{\Delta S_1(\omega_6)}{\Delta S_2(\omega_6)}.$$

We denote by $\Im(\tilde{A})$ the image of a mapping $\tilde{A}$. There exists statistical arbitrage if $\Im(\tilde{A}) \cap \mathbb{R}^4_{>0} \neq \emptyset$. The linear subspace spanned by $\tilde{A}$ is given by

$$\alpha \begin{pmatrix} 0 \\ 0 \\ B_1 \\ B_2 \end{pmatrix} + \beta \begin{pmatrix} \Delta S_2(\omega_2) \\ 0 \\ \Delta S_2(\omega_1)\nu_1 \\ \Delta S_2(\omega_3)\nu_2 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ \Delta S_2(\omega_6) \\ \Delta S_2(\omega_4) \\ \Delta S_2(\omega_5) \end{pmatrix}, \tag{32}$$

with $\alpha, \beta, \gamma \in \mathbb{R}$. Assume this space meets $\mathbb{R}^4_{\geq 0}$. Then it follows from the condition $\beta \Delta S_2(\omega_2) = \beta(s_2^{++} - s_1^{+}) \geq 0$ that $\beta \geq 0$. Similarly, $\gamma \leq 0$ because $\Delta S_2(\omega_6) = s_2^{--} - s_1^{-} < 0$. Summing up the third and fourth coordinate from (32) we get

$$\begin{aligned} \alpha \Bigg(& \nu_1 \left(\Delta S_1(\omega_1) - \Delta S_2(\omega_1) \frac{\Delta S_1(\omega_2)}{\Delta S_2(\omega_2)} \right) + \nu_2 \left(\Delta S_1(\omega_3) - \Delta S_2(\omega_3) \frac{\Delta S_1(\omega_2)}{\Delta S_2(\omega_2)} \right) \\ & + \frac{\Delta S_1(\omega_6)}{\Delta S_2(\omega_6)} \Big(- \Delta S_2(\omega_4) - \Delta S_2(\omega_5) \Big) + \Delta S_1(\omega_4) + \Delta S_1(\omega_5) \Bigg) \\ & + \gamma \left(\Delta S_2(\omega_4) + \Delta S_2(\omega_5) \right) + \beta \left(\Delta S_2(\omega_1)\nu_1 + \Delta S_2(\omega_3)\nu_2 \right). \end{aligned} \tag{33}$$

Choosing $\nu_1 = -\frac{\Delta S_2(\omega_3)}{\Delta S_2(\omega_1)} \nu_2$, we obtain that $\beta \left(\Delta S_2(\omega_1)\nu_1 + \Delta S_2(\omega_3)\nu_2 \right) = 0$, such that the last term in the above equation vanishes. As we assumed that the space spanned by (32) meets $\mathbb{R}^4_{\geq 0}$ it must also hold true that (33) ≥ 0. For

$$\nu_2 < \frac{\frac{\Delta S_1(\omega_6)}{\Delta S_2(\omega_6)} (\Delta S_2(\omega_4) + \Delta S_2(\omega_5)) - \Delta S_1(\omega_4) - \Delta S_1(\omega_5)}{\Delta S_1(\omega_3) - \Delta S_1(\omega_1) \frac{\Delta S_2(\omega_3)}{\Delta S_2(\omega_1)}} = \Gamma_2$$

the coefficient of α in (33) is negative. Together with $\gamma \leq 0$ and $\Delta S_2(\omega_4), \Delta S_2(\omega_5) > 0$ by assumption this choice of ν_2 results in $\alpha \leq 0$ in order to obtain (33) ≥ 0. On the other hand, if we claim

$$\nu_2 > \frac{-\Delta S_1(\omega_5) + \Delta S_2(\omega_5) \frac{\Delta S_1(\omega_6)}{\Delta S_2(\omega_6)}}{\Delta S_1(\omega_3) - \Delta S_2(\omega_3) \frac{\Delta S_1(\omega_2)}{\Delta S_2(\omega_2)}} = \Gamma_1$$

it follows that $B_2 > 0$ and it results for the fourth coordinate of (32) that $\alpha B_2 + \beta \Delta S_2(\omega_3)\nu_2 + \gamma \Delta S_2(\omega_5) \leq 0$. Hence $\Im(\tilde{A}) \cap \mathbb{R}^4_{>0} = \emptyset$. It remains to prove that $\Gamma_1 < \Gamma_2$ and that there is no statistical arbitrage for $\nu_2 = \Gamma_2$, which is verified analogously. □

Proof of Proposition 4.1. "⇒" If $\det(A) \neq 0$ we choose for example $\xi := A^{-1}1$ and have found an arbitrage opportunity.

"⇐" On the other hand, if $\det(A) = 0$ there still might be an arbitrage opportunity if the image of A intersects with the positive subspace of $\mathbb{R}^3$, i.e. if $\Im(A) \cap \mathbb{R}^3_{>0} \neq \emptyset$. To show that this is not the case we change the basis for the mapping A and substitute the vector in the first column. This leads to a matrix $\tilde{A}$,

$$\tilde{A} = \begin{pmatrix} 0 & \Delta S_2(\omega_1) & 0 \\ 0 & 0 & \Delta S_2(\omega_4) \\ B & \Delta S_2(\omega_2)q & \Delta S_2(\omega_3) \end{pmatrix},$$

where

$$B = q\big(\Delta S_1(\omega_1) - \frac{\Delta S_1(\omega_1)}{\Delta S_2(\omega_1)} \Delta S_2(\omega_2)\big) + \Delta S_1(\omega_3) - \frac{\Delta S_1(\omega_3)}{\Delta S_2(\omega_4)} \Delta S_2(\omega_3).$$

Hence, $0 = \det(A) = B\Delta S_2(\omega_1)\Delta S_2(\omega_4)$ leading to $B = 0$. In this case the linear subspace spanned by $\tilde{A}$ is given by

$$\alpha \begin{pmatrix} \Delta S_2(\omega_1) \\ 0 \\ q\Delta S_2(\omega_2) \end{pmatrix} + \beta \begin{pmatrix} 0 \\ \Delta S_2(\omega_4) \\ \Delta S_2(\omega_3) \end{pmatrix}, \tag{34}$$

with $\alpha, \beta \in \mathbb{R}$. Because $\Delta S_2(\omega_1) > 0$ we need $\alpha \geq 0$ to have arbitrage opportunities. Similar we need to have $\beta \leq 0$ because of $\Delta S_2(\omega_4) < 0$ by assumption. But, as $\Delta S_2(\omega_2) < 0$ and $\Delta S_2(\omega_3) > 0$, we obtain for the third coordinate that $\alpha q \Delta S_2(\omega_2) + \beta \Delta S_2(\omega_3) \leq 0$ and hence $\Im(A) \cap \mathbb{R}^3_{>0} = \emptyset$, which prooves the first part. For the second part, we simply compute $\det(A)$ from Equation (13). $\square$

Proof of Proposition 5.1. Following Definition 2.1 the strategy $\boldsymbol{\psi}$ is a statistical $\tilde{\mathscr{G}}$-arbitrage strategy if the following holds

$$\begin{aligned} \psi_1 \Delta S_1(\omega_1) + \psi_2^+ \Delta S_2(\omega_1) + \psi_3^{++} \Delta S_3(\omega_1) &\geq 0 \\ \psi_1 \Delta S_1(\omega_4) + \psi_2^- \Delta S_2(\omega_4) &\geq 0 \\ \psi_1 \Delta S_1(\omega_2) P(\omega_2) + \psi_2^+ \Delta S_2(\omega_2) P(\omega_2) + \psi_1 \Delta S_1(\omega_3) P(\omega_3) + \psi_2^- \Delta S_2(\omega_3) P(\omega_3) &\geq 0, \\ \psi_1 \Delta S_1(\omega_5) + \psi_2^+ \Delta S_2(\omega_5) + \psi_3^{++} \Delta S_3(\omega_5) &\geq 0 \end{aligned} \tag{35}$$

and, in addition, at least one of the inequalities is strict. We extend the setting from Lemma 4.2. First, we let

$$\tilde{A} = \begin{pmatrix} \Delta S_1(\omega_1) & \Delta S_2(\omega_1) & 0 & \Delta S_3(\omega_1) \\ \Delta S_1(\omega_4) & 0 & \Delta S_2(\omega_4) & 0 \\ q\Delta S_1(\omega_2) + \Delta S_1(\omega_3) & q\Delta S_2(\omega_2) & \Delta S_2(\omega_3) & 0 \\ \Delta S_1(\omega_5) & \Delta S_2(\omega_5) & 0 & \Delta S_3(\omega_5) \end{pmatrix}.$$

Then Equations (35) are equivalent to $\tilde{A}\boldsymbol{\psi} \geq 0$. Note that $S_i(\omega_1) = S_i(\omega_5)$ for $i = 1, 2$ such that $\tilde{A}\boldsymbol{\psi} = \tilde{\mathbf{x}}$ with $\tilde{\mathbf{x}} = (x_1, \dots, x_4)^\top$ reveals

$$\psi_3^{++} = \frac{x_1 - x_4}{\Delta S_3(\omega_1) - \Delta S_3(\omega_5)}.$$

As for Lemma 4.2, we will consider the case where $\tilde{A}$ is invertible. Note that the three times three submatrix (upper left) of $\tilde{A}$ equals the matrix A from Equation (13). Then, denoting $\mathbf{x} = (x_1, x_2, x_3)^\top$,

$$\begin{pmatrix} \psi_1 \\ \psi_2^+ \\ \psi_2^- \end{pmatrix} = A^{-1}\mathbf{x} - A^{-1} \begin{pmatrix} \Delta S_3(\omega_1)\psi_3^{++} \\ 0 \\ 0 \end{pmatrix} = A^{-1}\mathbf{x} - \Delta S_3(\omega_1)\psi_3^{++}\boldsymbol{\gamma}$$

with vector $\boldsymbol{\gamma}$ from Equation (26). Up to now we where free to choose any $\tilde{\mathbf{x}} \in \mathbb{R}^4_{>0}$. If we choose $\mathbf{x} = \mathbb{1}_3$, then $\boldsymbol{\phi} = A^{-1}\mathbb{1}_3$ is the strategy computed in Lemma 4.2 and the result follows. $\square$

Appendix B. Computation of a specific example of Equation (31)

Consider a random variable $\xi \sim \mathcal{N}(\mu, \sigma^2)$, such that $\xi = \mu + \sigma\eta$ with standard normal η. Then it is easy to verify that

$$\begin{aligned} E[\xi \mathbb{1}_{\{\xi > \alpha\}}] &= E[\mu \mathbb{1}_{\{\eta > c_\alpha\}}] + \sigma E[\eta \mathbb{1}_{\{\eta > c_\alpha\}}] = \mu(1 - \Phi(c_\alpha)) + \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}c_\alpha^2}, \\ E[\xi^2 \mathbb{1}_{\{\xi > \alpha\}}] &= (\mu^2 + \sigma^2)(1 - \Phi(c_\alpha)) + \frac{1}{\sqrt{2\pi}}(2\mu\sigma + \sigma^2 c_\alpha) e^{-\frac{1}{2}c_\alpha^2}, \\ E[\xi \mathbb{1}_{\{\xi < \beta\}}] &= \mu\Phi(c_\beta) - \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}c_\beta^2}, \\ E[\xi^2 \mathbb{1}_{\{\xi < \beta\}}] &= (\mu^2 + \sigma^2)\Phi(c_\beta) - \frac{1}{\sqrt{2\pi}}(2\mu\sigma + \sigma^2 c_\beta) e^{-\frac{1}{2}c_\beta^2}. \end{aligned}$$

Now, we obtain $E[\phi_2(\xi_1)(S_2 - \xi_1) | S_2 = s] = E[\phi_2(\xi)(s - \xi)]$, where $\mu = \mu(s) = \rho(s - \mu_2)$ and $\xi \sim \mathcal{N}(\mu, \sigma^2)$. Using the specific form of ϕ_2 and the above computations gives the result.

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