Risk bounds with additional information on functionals of the risk vector

L. Rüschendorf*

November 29, 2017

Abstract

We consider the problem of determining risk bounds for the Value at Risk for risk vectors X where besides the marginal distributions also information on the distribution or on the expectation of some functionals $T_j(X)$, $1 \leqslant j \leqslant m$, is available. In particular this formulation includes the case where information on subgroup sums or maxima or on the correlations or covariances is available. Based on the method of dual bounds we obtain improved risk bounds compared to the marginal case. In some cases we obtain sharp bounds.

Key-words: risk bounds, Value-at-Risk, dependence uncertainty, Fréchet class

AMS 2010 Subject Classification: 91B30 (primary); 60E15 (secondary)

1 Introduction

Besides the marginal information on the risk $X = (X_1, \dots, X_n)$ a useful additional source for reducing dependence uncertainty is to use additional information on some functionals of the risk vector. This information may be available in insurance type hierarchical models as information on the aggregation of some branches of the company evaluated by statistical analysis. F.e. as additional information the distribution of some subgroup sums $\sum_{j \in I_i} X_j \sim Q_i$ might be given where $I_i \subset \{1, \dots, n\}$, $1 \leqslant i \leqslant m$ or the worst case distribution of some subgroups $\max_{j \in I_i} X_j \sim Q_i$ might be known. Alternatively, information on (some) correlations $\tau_{ij} = \operatorname{Corr}(X_i, X_j)$ or covariances $\sigma_{ij} = \operatorname{Cov}(X_i, X_j)$ might be available.

In more general terms let $T_i : \mathbb{R}^n \to \mathbb{R}^1$, $i \in K$, be a class of measurable real functions. Assume that the distribution Q_i of $T_i(X)$ is known for $i \in K$. Under this information our aim is to derive upper resp. lower risk bounds for the Value at Risk on the aggregated risk $\sum_{i=1}^n X_i$. This formulation includes for the case $T_i(X) = \sum_{j \in I_i} X_j$ resp. $T_i(X) =$

*Ludger Rüschendorf, University of Freiburg, Eckerstraße 1, 79104 Freiburg, Germany. (email: ruschen@stochastik.uni-freiburg.de).

$\max_{j \in I_i} X_j$ the situation described above. If we also allow higher dimensional functions $T_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$, then we can in this way also describe the case where higher dimensional marginals are known, considering $T_i(X) = (X_j)_{j \in I_i}$, where $\mathcal{E} = \{I_i,\ i \in K\}$ is a marginal system and $Q_i = P^{X_{I_i}}$ are the corresponding higher dimensional marginal distributions.

Deriving risk bounds under additional information on the dependence has been the subject of several recent papers. In particular we mention information on variance bounds in Bernard et al. (2015), dependence information in Bignozzi et al. (2015), Puccetti et al. (2017), Bernard et al. (2017), Rüschendorf and Witting (2017), Lux and Rüschendorf (2017), Lux and Papapantoleon (2017) and the assumption that the distribution is known on subsets in Bernard and Vanduffel (2015), Puccetti et al. (2016), Lux and Papapantoleon (2016). Some survey papers on this subject are available by Rüschendorf (2017a,b).

In this paper we extend the method of dual bounds as described in Puccetti and Rüschendorf (2013) and Embrechts et al. (2013) in the context of marginal information to the case where additional information is available. In several of the above mentioned papers in a first step exact dual representations of the VaR or max risk bounds have been derived, which however turn out to be too involved for determining solutions. Based on these results then in a second step relaxed versions of these duality results have been introduced which are accessible by analytical or numerical methods. We concentrate in this paper on the second step of getting usable bounds and do not elaborate on exact duality results. In this way we obtain a unified approach which yields improved risk bounds depending on the degree and quality of the additional information. In several cases we even get sharp bounds.

2 Functionals of the risk vector

Knowing higher order marginals P_I, $I \in \mathcal{E}$ implies that also the distribution of all functionals $T(X) = T_I(X_I)$ is known. This assumption is in applications often too strong. In this section we assume that for some real functionals $T_1, \dots, T_m$ the distribution of $T_i(X)$, $1 \leqslant i \leqslant m$ is known.

We call this the *(DF)-assumption* for functionals T_i:

$$T_i(X) \sim Q_i, \quad 1 \leqslant i \leqslant m \tag{2.1}$$

We also consider a weaker form of this assumption, the *(EF)-assumption*

$$ET_i(X) = a_i, \quad 1 \leqslant i \leqslant m, \tag{2.2}$$

and the corresponding *($EF^{\leqslant}$)-assumption*

$$ET_i(X) \leqslant a_i, \quad 1 \leqslant i \leqslant m. \tag{2.3}$$

The (DF)- and (EF)-assumptions are quite flexible and widely applicable and allow a unified derivation of upper and lower tail-risk bounds and, therefore, of reduced upper and lower VaR-bounds.

Several methods in the literature for VaR-reduction can be subsumed under this kind of functional assumptions. Assuming that $\mathrm{Var}(S_n) \leqslant \sigma^2$ as in Bernard et al. (2017) or a related higher moment assumptions $ES_n^k \leqslant c_k$ is exactly of the form (EF$^{\leqslant}$) for one functional $T_1(X) = (\sum_{i=1}^n X_i)^2$ resp. $(\sum_{i=1}^n X_i)^k$. Assuming that the distributions G_r of the weighted subgroup sums

$$Y_r = \sum_{j \in I_r} \frac{1}{\eta_j} X_j = T_r(X), \quad 1 \leqslant r \leqslant m, \tag{2.4}$$

with $\eta_j := |\{r;\ j \in I_r\}|$ and $\mathcal{E} = \{I_1, \dots, I_m\}$ are known, is of the form (DF) and is a weakening of the assumption of knowing the multivariate distributions F_I of X_I for $I \in \mathcal{E}$ as in Embrechts and Puccetti (2006), or Puccetti and Rüschendorf (2012). Similarly we can consider the case

$$T_r(X) = \max_{j \in I_r} X_j, \quad 1 \leqslant r \leqslant m, \tag{2.5}$$

of knowing the distributions of the subgroup maxima. Assuming knowledge of the covariances or correlations corresponds to (EF) in the case where

$$T_{i,j}(X) = X_i X_j, \quad 1 \leqslant i < j \leqslant n \tag{2.6}$$

and $a_{i,j} = \sigma_{i,j} + \mu_i \mu_j$ are the corresponding mixed expectations.

We denote by $M^{\mathrm{DF}}, m^{\mathrm{DF}}$ resp. $M^{\mathrm{EF}}, m^{\mathrm{EF}}$ resp. $M^{\mathrm{EF}^{\leqslant}}, m^{\mathrm{EF}^{\leqslant}}$ the corresponding maximal resp. minimal tail risk probabilities and the corresponding VaR bounds by $\mathrm{Var}_\alpha^{\mathrm{MF}}$, etc. Under the (DF)-assumption we introduce dual problems of the form

$$\begin{aligned} U^{\mathrm{DF}}(s) &= \inf \Big\{ \sum_{i=1}^n \int f_i dF_i + \sum_{j=1}^m \int g_j dQ_j;\ \ (f_i, g_j) \in \overline{A}^{\mathrm{DF}}(s) \Big\} \\ \text{and} \quad I^{\mathrm{DF}}(s) &= \sup \Big\{ \sum_{i=1}^n \int f_i dF_i + \sum_{j=1}^m \int g_j dQ_j;\ \ (f_i, g_i) \in \underline{A}^{\mathrm{DF}}(s) \Big\} \end{aligned} \tag{2.7}$$

where $\overline{A}^{\mathrm{DF}}(s) = \Big\{ (f_i, g_j);\ \ \sum_{i=1}^n f_i(x_i) + \sum_{i=1}^m g_i \circ T_j(x) \geqslant 1_{[s,\infty)} \Big(\sum_{j=1}^n X_j \Big) \Big\}$ and

$\underline{A}^{\mathrm{DF}}(s) = \Big\{ (f_i, g_j);\ \ \sum_{i=1}^n f_i(x_i) + \sum_{i=1}^m g_i \circ T_j(x) \leqslant 1_{[s,\infty)} \Big(\sum_{j=1}^n X_j \Big) \Big\}$, and $F_i \sim X_i$.

Under several regularity conditions strong duality theorems for these functionals can be proved. For some examples see f.e. Rüschendorf (2017a). We formulate the simple to verify upper and lower bounds properties of these dual functionals and abstain from discussing the more involved strong duality results.

Proposition 2.1 (Upper and lower bounds under (DF))**.** *Assume that the risk vector satisfies assumption (DF) in* (2.1) *for functionals* $T_1, \dots, T_m$*. Then the following improved tail risk bounds hold:*

$$M^{DF}(s) \leqslant U^{DF}(s) \quad \textit{and} \quad m^{DF}(s) \geqslant I^{DF}(s). \tag{2.8}$$

Proof. By assumption (DF) holds for any $(f_i, g_j) \in \overline{A}^{\mathrm{DF}}(s)$, i.e. $\sum_{i=1}^n f_i(x_i) + \sum_{j=1}^m g_j \circ T_j(x) \geqslant 1_{[s,\infty)}(\sum_{j=1}^n x_j)$,

$$\begin{aligned} P\Big(\sum_{i=1}^n X_i \geqslant s\Big) &\leqslant E\sum_{i=1}^n f_i(X_i) + \sum_{j=1}^m E g_j \circ T_j(X) \\ &= \sum_{i=1}^n \int f_i dF_i + \sum_{j=1}^m \int g_j dQ_j. \end{aligned}$$

This implies taking sup on the left-hand side and inf on the right-hand side, $M^{\mathrm{DF}}(s) \leqslant U^{\mathrm{DF}}(s)$. The inequality $m^{\mathrm{DF}} \geqslant I^{\mathrm{DF}}$ follows similarly. □

In some cases it may be useful to relax the dual problems by omitting the simple marginal information, i.e. considering admissible dual functions of the form $(0, g_j)$. Define

$$\begin{aligned} \widetilde{U}^{\mathrm{DF}}(s) &= \inf\Big\{\sum_{j=1}^m \int g_j dQ_j;\ \sum_{j=1}^m g_j(x) \geqslant 1_{[s,\infty)}\Big(\sum_{j=1}^n x_j\Big)\Big\} \\ \text{and}\quad \widetilde{I}^{\mathrm{DF}}(s) &= \sup\Big\{\sum_{j=1}^m \int g_j dQ_j;\ \sum_{j=1}^m g_j(x) \leqslant 1_{[s,\infty)}\Big(\sum_{j=1}^n x_j\Big)\Big\}. \end{aligned} \tag{2.9}$$

Corollary 2.2 (Relaxed upper and lower bounds under (DF))**.** *Under assumption (DF) for $T_1, \dots, T_m$ holds:*

$$M^{DF}(s) \leqslant \widetilde{U}^{DF}(s) \quad \text{and} \quad m^{DF}(s) \geqslant \widetilde{I}^{DF}(s). \tag{2.10}$$

Proof. Corollary 2.2 follows from Proposition 2.1 noting that by definition of $\widetilde{U}^{\mathrm{DF}}$ and $\widetilde{I}^{\mathrm{DF}}$ it holds that

$$U^{\mathrm{DF}}(s) \leqslant \widetilde{U}^{\mathrm{DF}}(s) \quad \text{and} \quad I^{\mathrm{DF}}(s) \geqslant \widetilde{I}^{\mathrm{DF}}(s). \qquad \square$$

In some cases the sharpness of the bounds can be seen directly and the dual bounds in (2.8) and (2.10) can be reduced to simple marginal bounds. We consider the subgroup sum case in (2.4) for a marginal system $\mathcal{E} = \{I_1, \dots, I_m\}$ and the weighted sum

$$T_r(x) = \sum_{j \in I_r} \frac{1}{\eta_j} X_j =: Y_r. \tag{2.11}$$

Here η_j counts the number of subsets which have j as an element. In the case of non-overlapping sets $\{I_r\}$ with $\bigcup_{r=1}^m I_r = \{1, \dots, n\}$ holds $\eta_j = 1$, $\forall j$ and the weighted sum Y_r is identical to the partial sum over subgroup I_r.

Denote $H_r = F_{Y_r}$ the partial sum distribution of Y_r and $\mathcal{H} = \mathcal{F}(H_1, \dots, H_m)$ the corresponding simple marginal systems. Under assumption (DF) the distributions H_r of Y_r are known and we obtain:

Theorem 2.3 (Upper and lower bounds with partial sum information)**.** *Let the risk vector X satisfy assumption (DF) for the partial sum functionals in* (2.11)*. Then:*

a) $$U^{DF}(s) \leqslant \widetilde{U}^{DF}(s) \quad \text{and} \quad I^{DF}(s) \geqslant \widetilde{I}^{DF}(s)$$

b)
$$\begin{aligned} M_{\mathcal{E}}(s) \leqslant M^{DF}(s) \leqslant M_{\mathcal{H}}(s) = \widetilde{U}^{DF}(s) \\ \text{and} \quad m_{\mathcal{E}}(s) \geqslant m^{DF}(s) \geqslant m_{\mathcal{H}}(s) = \widetilde{I}^{DF}(s) \end{aligned} \tag{2.12}$$

c) If the marginal system is non-overlapping, then
$$\begin{aligned} M_{\mathcal{E}}(s) = M^{DF}(s) = M_{\mathcal{H}}(s) = \widetilde{U}^{DF}(s) \\ \text{and} \quad m_{\mathcal{E}}(s) = m^{DF}(s) = m_{\mathcal{H}}(s) = \widetilde{I}^{DF}(s). \end{aligned} \tag{2.13}$$

Proof. a), b) The proof of a) and b) follows by combining Proposition 2.1 and Corollary 2.2 with the arguments used in the proof of Theorem 3.5 in Puccetti and Rüschendorf (2012) for the inequality $M_{\mathcal{E}}(s) \leqslant M_{\mathcal{H}}(s)$. In particular note that $\sum_{i=1}^n X_i = \sum_{r=1}^m Y_r$ and that by assumption (DF) we have that $F_{Y_r} = H_r$. The equality $M_{\mathcal{H}}(s) = \widetilde{U}^{DF}(s)$ is the classical strong duality for simple marginal systems.

c) From b) we have the inequality
$$M_{\mathcal{E}}(s) \leqslant M_{\mathcal{H}}(s).$$
Conversely, if $Y = (Y_1, \dots, Y_m)$ is any vector with distribution function $F_Y \in \mathcal{H}$, then by a classical result on stochastic equations there exist $X_{I_r} \sim F_{I_r}$ such that $\sum_{j \in I_r} X_j = Y_r$ a.s., $1 \leqslant r \leqslant m$. This implies the converse inequality
$$M_{\mathcal{H}}(s) \leqslant M_{\mathcal{E}}(s).$$

Since for the simple marginal system the strong duality theorem holds we obtain
$$M_{\mathcal{H}}(s) = \widetilde{U}^{\mathrm{DF}}(s)$$
and thus equalities in (2.12) are obtained. The case of the lower bounds is similar. □

In consequence of Theorem 2.3 we only need the weaker assumption (DF) of knowledge of the distribution of the subgroup sums in order to derive the same bounds as in Theorems 3.3 and 3.5 in Puccetti and Rüschendorf (2012) given there under the stronger assumption of knowledge of the higher order marginals.

Under the (EF)- resp. (EF$^{\leqslant}$)-assumption for functionals $T_1, \dots, T_m$ we similarly to the (DF)-case introduce dual functionals
$$U^{\mathrm{EF}}(s) = \inf\Big\{ \sum_{i=1}^n \int f_i dF_i + \sum_{j=1}^m \lambda_j a_j;\ \ (f_i, \lambda_j) \in \overline{A}^{\mathrm{EF}}(s) \Big\} \tag{2.14}$$
$$\text{and} \quad I^{\mathrm{EF}}(s) = \sup\Big\{ \sum_{i=1}^n \int f_i dF_i + \sum_{j=1}^m \lambda_j a_j;\ \ (f_i, \lambda_j) \in \underline{A}^{\mathrm{EF}}(s) \Big\} \tag{2.15}$$
and, similarly, in the inequality case $U^{\mathrm{EF}^{\leqslant}}, I^{\mathrm{EF}^{\leqslant}}$, where
$$\overline{A}^{\mathrm{EF}}(s) = \Big\{ (f_i, \lambda_j);\ \ f_i \in L^1(F_i), \lambda_j \in \mathbb{R}, \sum_{i=1}^n f_i(x_i) + \sum_{j=1}^m \lambda_j T_j(x) \geqslant 1_{[s,\infty)}\Big(\sum_{j=1}^n x_j \Big) \Big\},$$
$$\overline{A}^{\mathrm{EF}^{\leqslant}}(s) = \Big\{ (f_i, \lambda_j);\ \ f_i \in L^1(F_i), \lambda_j \in \mathbb{R}_+, \sum_{i=1}^n f_i(x_i) + \sum_{j=1}^m \lambda_j T_j(x) \leqslant 1_{[s,\infty)}\Big(\sum_{j=1}^n x_j \Big) \Big\}$$

and $\underline{A}^{\mathrm{EF}}(s)$ and $\underline{A}^{\mathrm{EF}^{\leqslant}}(s)$ are defined similarly.

Proposition 2.4 (Upper and lower bounds under (EF) resp. (EF$^{\leqslant}$))**.** *Assume that the risk vector satisfies:*

a) Assumption (EF) for the functionals $T_1, \dots, T_m$, then

$$M^{EF}(s) \leqslant U^{EF}(s) \quad \text{and} \quad m^{EF}(s) \geqslant I^{EF}(s). \tag{2.16}$$

b) Assumption (EF$^{\leqslant}$) for $T_1, \dots, T_m$, then

$$M^{EF^{\leqslant}}(s) \leqslant U^{EF^{\leqslant}}(s) \quad \text{and} \quad m^{EF^{\leqslant}}(s) \geqslant I^{EF^{\leqslant}}(s). \tag{2.17}$$

Proof. a) This follows as in Proposition 2.1 using that for $(f_i, \lambda_j) \in \overline{A}^{\mathrm{EF}}(s)$

$$\sum_{i=1}^n f_i(x_i) + \sum_{j=1}^m \lambda_j T_j(x) \geqslant 1_{[s,\infty)}\Big(\sum_{i=1}^n X_i\Big). \tag{2.18}$$

b) Under (EF$^{\leqslant}$) we have $ET_j(X) \leqslant a_j$ and, therefore, since $\lambda_j \geqslant 0$ we have $E\lambda_j T_j(X) \leqslant \lambda_j a_j$ and thus the inequality (2.18) implies

$$P\Big(\sum_{j=1}^n X_j \geqslant s\Big) \leqslant \sum_{i=1}^n \int f_i dF_i + \sum_{j=1}^m \lambda_j a_j. \qquad \square$$

In some cases it may be useful to omit the marginal information and use relaxed duals as in (2.9) and Corollary 2.2. Define

$$\widetilde{M}^{\mathrm{EF}}(s) = \inf\Big\{\sum_{j=1}^m \lambda_j a_j;\ \ \lambda_j \in \mathbb{R}, \sum_{j=1}^m \lambda_j T_j(x) \geqslant 1_{[s,\infty)}\Big(\sum_{i=1}^n X_i\Big)\Big\} \tag{2.19}$$

and, similarly, $\widetilde{I}^{\mathrm{EF}}(s)$, $\widetilde{U}^{\mathrm{EF}^{\leqslant}}(s)$ and $\widetilde{I}^{\mathrm{EF}^{\leqslant}}(s)$.

Corollary 2.5 (Relaxed upper and lower bounds under (EF) resp. (EF$^{\leqslant}$))**.**

a) Under assumption (EF) for $T_1, \dots, T_m$ holds

$$M^{EF}(s) \leqslant \widetilde{U}^{EF}(s) \quad \text{and} \quad m^{EF}(s) \geqslant \widetilde{I}^{EF}(s) \tag{2.20}$$

b) Under assumption (EF$^{\leqslant}$) for $T_1, \dots, T_m$ holds

$$M^{EF^{\leqslant}}(s) \leqslant \widetilde{U}^{EF^{\leqslant}}(s) \quad \text{and} \quad m^{EF^{\leqslant}}(s) \geqslant \widetilde{I}^{EF^{\leqslant}}(s) \tag{2.21}$$

Remark 2.6. *The dual method to derive upper and lower bounds for the tail risks under the assumptions (EF), (EF$^{\leqslant}$) and (DF) has immediate extensions to derive upper and lower bounds for the expectation $E\varphi(X)$ of a function φ of the risk vector. We just have to change the admissible class of dual functions f.e. change $\overline{A}^{DF}(s)$ to*

$$\overline{A}^{DF}(\varphi) := \Big\{(f_i, g_j);\ \sum_{i=1}^n f_i(x_i) + \sum_{j=1}^m g_j(T_j(x)) \geqslant \varphi(x)\Big\}. \tag{2.22}$$

Denoting the maximal expectation of φ under (DF) by $M^{DF}(\varphi)$, we obtain similarly to Proposition 2.1:

$$M^{DF}(\varphi) \leqslant U^{DF}(\varphi) \quad \text{and} \quad m^{DF}(\varphi) \geqslant I^{DF}(\varphi). \tag{2.23}$$

Similar bounds are valid under (EF) and ($EF^{\leqslant}$).

We consider some applications of assumption (DF) with maxima or with subgroup maxima information. Let $\mathcal{E} = \{I_1, \dots, I_m\}$ be a non-overlapping system with $\bigcup_{j=1}^m I_j = \{1, \dots, n\}$ and consider the subgroup maxima

$$T_r(X) := \max_{j \in I_r} X_j. \tag{2.24}$$

Under assumption (DF) for $T_1, \dots, T_m$, i.e. knowing the distribution G_r of the subgroup maxima

$$G_r \sim T_r(X) \tag{2.25}$$

we obtain improved bounds for the distribution function or the survival function (tail risk) of the max in comparison to the case of marginal information only, dealt with in Rüschendorf (1980) (see also Corollary 2.20 in Rüschendorf (2013)).

Theorem 2.7 (Tail risk of max under subgroup max information)**.** *Let the risk vector X satisfy condition (DF) for the subgroup maxima $T_1, \dots, T_m$ in* (2.24)*, then*

$$\Big(\sum_{r=1}^m G_r(t) - (m-1) \Big)_+ \leqslant F_{\max_{1 \leqslant i \leqslant n} X_i}(t) \leqslant \min_{r \leqslant m} G_r(t) \tag{2.26}$$

and the bounds in (2.26) *are sharp.*

Proof. By definition of the subgroup maxima T_r we have the basic equality

$$\max_{1 \leqslant i \leqslant n} X_i = \max_{r=1,\dots,m} T_r(X) \tag{2.27}$$

. Since $T_r(X) \sim G_r$, we obtain from the Hoeffding-Fréchet bounds that the bounds in (2.27) are valid. Defining $Y_1, \dots, Y_m$ as maximally dependent vector with marginal distributions $G_1, \dots, G_m$ we obtain (see Rüschendorf (1980))

$$\max_{i \leqslant m} Y_i \sim \Big(\sum_{r=1}^m G_r(t) - (m-1) \Big)_+. \tag{2.28}$$

Similarly, considering $Z_1, \dots, Z_m$, the comonotonic vector with marginals G_i, the upper bound is attained

$$\max_{i \leqslant m} Z_i \sim \min_{r \leqslant m} G_r(t). \tag{2.29}$$

By a well-known result on stochastic equations it is possible to construct a vector X with marginals F_i such that a.s. $\max_{j \in I_r} X_j = Y_r$ and similarly it is possible to construct a vector X with marginals F_i such that a.s. $\max_{j \in I_r} X_j = Z_r$. This implies that the bounds in (2.26) are sharp. □

Remark 2.8. *a) By* (2.26) *holds for the maximal tail risk* $\overline{F}_{\max\limits_{i\leqslant n} X_i}(t)$*:*

$$\max_{j\leqslant m} \overline{G}_r(t) \leqslant \overline{F}_{\max\limits_{i\leqslant n} X_i}(t) \leqslant \min\Big\{\Big(m - \sum_{r=1}^m G_r(t)\Big)_+, 1\Big\}. \tag{2.30}$$

This is an improvement over the sharp simple marginal bound

$$\max_{1\leqslant i\leqslant n} \overline{F}_i(t) \leqslant \overline{F}_{\max\limits_{1\leqslant n} X_i}(t) \leqslant \min_{i\leqslant n}\Big(n - \sum_{i=1}^n F_i(t), 1\Big). \tag{2.31}$$

b) Theorem 2.7 also results from an application of the relaxed dual bounds as in Corollary 2.5 based on the inequality

$$\max_{1\leqslant i\leqslant n} X_i \leqslant \inf_{v\in\mathbb{R}^m} \sum_{r=1}^m \big(v_r + (T_r(X) - v_r)_+\big) \tag{2.32}$$

and noting, that the upper bound is attained for the maximally dependent vector $T_r(X) = Y_r$*,* $1 \leqslant r \leqslant m$*. This gives the lower bound in* (2.33) *while the upper bound is a direct consequence of the stochastic ordering result*

$$T_r(X) \geqslant_{st} Z_r, \ \ 1 \leqslant r \leqslant m. \tag{2.33}$$

c) For the upper tail risk of the aggregated sum given the assumption (EF) for the subgroup max functionals $T_r(X) = \max_{j\in I_r} X_j$ *by the reduced form of the dual functionals it is indicated to consider inequalities of the form*

$$\sum_{i=1}^n x_i \leqslant \sum_{r=1}^m \alpha_r \max_{j\in I_r} x_j. \tag{2.34}$$

Assuming that $x_i \geqslant 0$ *this inequality implies* $\alpha_r \geqslant n_r = |I_r|$ *and as consequence this implies the tail risk bound for the sum*

$$M^{EF}(s) \leqslant M_{\mathcal{H}}(s), \tag{2.35}$$

where $\mathcal{H} = \{H_1, \dots, H_m\}$*,* $H_r(t) = G_r\Big(\frac{t}{n_r}\Big)$*. The upper bound in* (2.35) *can be evaluated by the RA-algorithm but it seems typically to be too rough based on the rough inequality in* (2.34)*.*

The tail risks problem in Remark 2.8 c) with maximal subgroup information seems to be better dealable with by the method based on knowledge of a distribution function on a subset as dealt with in Puccetti et al. (2016) and Lux and Papapantoleon (2017). Note that knowing the distribution G_r of $Y_r = \max_{j\in I_r} X_j$ amounts to knowing the distribution function $F_{X_{I_r}} = F^{(r)}$ on the subset $S = \{(t, \dots, t);\ t \in \mathbb{R}\}$ since

$$F_{X_{I_r}}((t, \dots, t)) = P(X_j \leqslant t;\ j \in I_r) = P(\max_{j\in I_r} X_j \leqslant t) = G_r(t). \tag{2.36}$$

This implies by the improved Hoeffding-Fréchet bounds in Puccetti et al. (2017) and Lux and Papapantoleon (2017) that

$$\underline{F}_r^S(x) \leqslant F^{(r)}(x) \leqslant \overline{F}_r^S(x), \quad x \in \mathbb{R}^{I_r}, \tag{2.37}$$

where

$$\overline{F}_r^S(x) = \min\Big(\min_{j\in I_r} F_j(x_j), \inf_{t\in\mathbb{R}}\Big\{G_r(t) + \sum_{j\in I_r}(F_j(x_j) - F_j(t))_+\Big\}\Big)$$

$$\text{and}\quad \underline{F}_r^S(x) = \max\Big(0, \sum_{j\in I_r} F_j(x_j) - (n_r - 1), \sup_t \Big\{G_r(t) + \sum_{j\in I_r}(F_j(t) - F_j(x_j))_+\Big\}\Big). \tag{2.38}$$

The bounds in (2.37) imply that the distribution function $F = F_X$ of the risk vector is bounded by

$$\underline{F}^S(x) := \Big(\sum_{r=1}^m \underline{F}_r^S(x_{I_r}) - (n-1)\Big)_+ \leqslant F(x) \leqslant \min_{r=1,\dots,m} \overline{F}_r^S(x_{I_r}) =: \overline{F}^S(x). \tag{2.39}$$

These estimates allow to apply the method of improved standard bounds. This method was introduced in Williamson and Downs (1990), Denuit et al. (1999), Embrechts et al. (2003), Rüschendorf (2005), Embrechts and Puccetti (2006), see also (Puccetti and Rüschendorf 2012, Theorem 3.1).

Theorem 2.9 (Tail risk of sum under subgroup max information)**.** *Let the risk vector X satisfy condition (DF) for the subgroup maxima $T_1, \dots, T_m$ in* (2.24)*, then*

$$P\Big(\sum_{i=1}^n X_i \leqslant s\Big) \geqslant \bigvee \underline{F}^S(s), \tag{2.40}$$

where $\bigvee$ denotes the sup-convolution.

Remark 2.10. *Similarly we get a lower bound for the tail risk of the sum. Denoting the survival function of $F^{(r)}$ by $\widehat{F}^r$ and defining*

$$\widehat{F}_r^S(x) = \max\Big(0, \sum_{j\in I_r} \overline{F}_j(x_j) - (n_r - 1), \sup_t \Big(G_r(t) - \sum_{j\in I_r}\big(\overline{F}_j(t) - \overline{F}_j(x_j)\big)_+\Big)\Big)$$

we obtain

$$P\Big(\sum_{i=1}^n X_i \geqslant s\Big) \geqslant \bigvee \widehat{F}_r^S(s). \tag{2.41}$$

We next consider an example for the application of the (DF) condition to an optimization problem for stop loss premia.

Example 2.11 (Stop loss premia for a portfolio with additional sum information)**.** *Assume that $X = (X_1, X_2)$ is a risk vector with marginals $X_1 \sim F_1$, $X_2 \sim F_2$ and assume that the distribution of $T(X) = X_1 + X_2$ is known to be G_1, i.e. we make the (DF) assumption for $T_1 = T$. Our aim is to determine under this condition the maximum stop loss premium for the portfolio $2X_1 + X_2$, i.e. to determine $M^{DF}(\varphi_s)$ for $\varphi_s(x) = (2x_1 + x_2 - s)_+$. With the mean excess functions defined for a random variable X by $\pi_X(t) = E(X - t)_+$ the problem can be written in the form*

$$\pi_{2X_1+X_2}(t) = \max \tag{2.42}$$

under assumption (DF), as specified above.

Note that for all $a_1, a_2, a_3 \geqslant 0$ *with*

$$a_1 + a_3 = 2 \text{ and } a_2 + a_3 = 1 \text{ and for all } u = (u_i) \text{ with } u_1 + u_2 + u_3 = s \tag{2.43}$$

we have

$$(2x_1 + x_1 - s)_+ \leqslant (a_1 x_1 - u_1)_+ + (a_2 x_2 - u_2)_+ + (a_3(x_1 + x_2) - u_3)_+. \tag{2.44}$$

This inequality implies taking expectations and infima

$$\begin{aligned} M^{DF}(\varphi_s) &= \sup\{E(2X_1 + X_2 - s)_+;\ \ X \text{ satisfies (DF)}\} \\ &\leqslant \inf\Big\{a_1 \pi_{X_1}\Big(\frac{u_1}{a_1}\Big) + a_2 \pi_{X_2}\Big(\frac{u_2}{a_2}\Big) + a_3 \pi_{X_1+X_2}\Big(\frac{u_3}{a_3}\Big);\ \ a, u \text{ satisfying (2.43)}\Big\} \end{aligned} \tag{2.45}$$

By assumption DF for T the excess functions are known. This dual problem can be solved for distributions with analytical form of the mean excess functions involved.

In the following application for the method of dual bounds we consider the case where additional to the marginals also the covariances

$$\sigma_{ij} = \operatorname{Cov}(X_i, X_j) = EX_iX_j - \mu_i\mu_j, \quad \mu_i = EX_i, \tag{2.46}$$

are specified. This corresponds to assumption (EF) for $T_{ij}(X) = X_iX_j$ with $s_{ij} = EX_iX_j = \sigma_{ij} + \mu_i\mu_j$. From Proposition 2.4 we obtain the improved upper bounds formulated here for a function φ of the risk vector (as in Remark 2.6).

Theorem 2.12 (Risk bound with covariance information)**.** *a) Let the risk vector X satisfy additionally the moment information (EF) $EX_iX_j = s_{ij}$, $1 \leqslant i \leqslant j \leqslant n$, then for a risk function φ holds*

$$\begin{aligned} M^{EF}(\varphi) \leqslant U^{EF}(\varphi) = \inf\Big\{ &\sum_{i=1}^n \int f_i dF_i + \sum_{i,j=1}^n \alpha_{ij} s_{ij};\quad f_i \in L^1(F_i), \alpha_{ij} \in \mathbb{R}, \\ &\varphi \leqslant \sum_{i=1}^n f_i(x_i) + \sum \alpha_{ij} x_i x_j \Big\} \end{aligned} \tag{2.47}$$

b) Under (EF$^{\leqslant}$) the inequality (2.47) *holds with* $\alpha_{ij} \in \mathbb{R}_+$.

Remark 2.13. *a) For certain classes of functions φ the exact duality in* (2.47) *is stated in Rüschendorf (2017a).*

b) Considering as in Bernard et al. (2015) $\varphi = \mathbb{1}_{\{\sum_{i=1}^n x_i \geqslant s\}}$, the tail risk of the sum functional and assuming that besides the marginals F_i it is known that

$$ES_n^2 \leqslant s^2 \tag{2.48}$$

or, equivalently, $\operatorname{Var} S_n \leqslant \alpha^2 = s^2 - \mu^2$, $\mu = ES_n$, *then the dual corresponding to* (2.47) *simplifies to the form*

$$\begin{aligned} U^{EF^{\leqslant}}(s) = \inf\Big\{ &\sum_{i=1}^n \int f_i dF_i + \alpha s^2;\quad \alpha \geqslant 0, f_i \in L^1(F_i), \\ &\mathbb{1}_{\{\sum_{i=1}^n X_i \geqslant s\}} \leqslant \sum_{i=1}^n f_i(x_i) + \alpha\Big(\sum_{i=1}^n x_i\Big)^2 \Big\} \end{aligned} \tag{2.49}$$

In Bernard et al. (2017) good upper bounds for this case are given. In comparison (2.49) *gives theoretical sharp upper bounds which can be evaluated however only in strongly relaxed form.*

c) ***Model independent price bounds.*** *In a similar way the method of dual bounds in this chapter also applies to various other types of constraints. For robust model independent price bounds in recent years dual representations with martingale constraints have been developed (see Acciaio et al. (2016) and Beiglböck et al. (2013)). These constraints are due to the fact that reasonable pricing measures have the martingale property. The dual method in this chapter can be extended to infinitely many constraints to deal with the problem of determining robust model independent price bounds i.e. with model independent price bounds based solely on the martingale constraint additional to the marginal structure over time.*

References

B. Acciaio, M. Beiglböck, F. Penkner, and W. Schachermayer. A model-free version of the fundamental theorem of asset pricing and the super replication theorem. *Mathematical Finance*, 26:233–251, 2016.

M. Beiglböck, P. Henry-Labordère, and F. Penkner. Model-independent bounds for option prices – a mass transport approach. *Finance Stoch.*, 17(3):477–501, 2013.

C. Bernard and S. Vanduffel. A new approach to assessing model risk in high dimensions. *Journal of Banking and Finance*, 58:167–178, 2015.

C. Bernard, L. Rüschendorf, and S. Vanduffel. Value-at-risk bounds with variance constraints. *Journal of Risk and Insurance*, 2015. doi: 10.111/jori.12108.

C. Bernard, L. Rüschendorf, S. Vanduffel, and R. Wang. Risk bounds for factor models. *Finance and Stochastics*, 3:631–659, 2017.

V. Bignozzi, G. Puccetti, and L. Rüschendorf. Reducing model risk via positive and negative dependence assumptions. *Insur. Math. Econ.*, 61:17–26, 2015.

M. Denuit, J. Genest, and É. Marceau. Stochastic bounds on sums of dependent risks. *Insur. Math. Econ.*, 25(1):85–104, 1999.

P. Embrechts and G. Puccetti. Bounds for functions of multivariate risks. *J. Multivariate Anal.*, 97(2):526–547, 2006.

P. Embrechts, A. Höing, and A. Juri. Using copulae to bound the Value-at-Risk for functions of dependent risks. *Finance and Stochastics*, 7(2):145–167, 2003.

P. Embrechts, G. Puccetti, and L. Rüschendorf. Model uncertainty and VaR aggregation. *Journal of Banking and Finance*, 37(8):2750–2764, 2013.

T. Lux and A. Papapantoleon. Model-free bounds on Value-at-Risk using partial dependence information. Preprint, 2016.

T. Lux and A. Papapantoleon. Improved Fréchet–Hoeffding bounds on d-copulas and applications in model-free finance. *Annals of Applied Probability*, forthcoming, 2017.

T. Lux and L. Rüschendorf. VaR bounds with two-sided dependence information on the copula. Preprint, 2017.

G. Puccetti and L. Rüschendorf. Bounds for joint portfolios of dependent risks. *Statistics & Risk Modeling*, 29:107–132, 2012.

G. Puccetti and L. Rüschendorf. Sharp bounds for sums of dependent risks. *Journal of Applied Probability*, 50(1):42–53, 2013.

G. Puccetti, L. Rüschendorf, and D. Manko. VaR bounds for joint portfolios with dependence constraints. *Dependence Modeling*, 4:368–381, 2016.

G. Puccetti, L. Rüschendorf, D. Small, and S. Vanduffel. Reduction of value-at-risk bounds via independence and variance information. *Scand. Actuarial J.*, 3:245–266, 2017.

L. Rüschendorf. Inequalities for the expectation of Δ-monotone functions. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, 54:341–354, 1980.

L. Rüschendorf. Stochastic ordering of risks, influence of dependenc, and a.s. constructions. In N. Balakrishnan, I. G. Bairamov, and O. L. Gebizlioglu, editors, *Advances on Models, Characterization and Applications*, pages 19–56. Chapman and Hall/CRC, 2005.

L. Rüschendorf. *Mathematical Risk Analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios.* Springer, 2013.

L. Rüschendorf. Improved Hoeffding–Fréchet bounds and applications to VaR estimates. In M. Úbeda Flores et al., editors, *Copulas and Dependence Models with Applications*, pages 181–202. Springer, 2017a.

L. Rüschendorf. Risk bounds and partial dependence information. In D. Ferger, W. González Manteiga, T. Schmidt, and J.-L. Wang, editors, *From Statistics to Mathematical Finance. Festschrift in Honour of Winfried Stute*, pages 345–366. Springer, 2017b.

L. Rüschendorf and J. Witting. VaR bounds in models with partial dependence information on subgroups. *Dependence Modeling*, 5(1):59–74, 2017.

R. C. Williamson and T. Downs. Probabilistc arithmetic. i. numerical methods for calculationg convolutions and dependency bounds. *International Journal of Approximate Reasoning*, 4 (2):89–158, 1990.