Risk bounds and partial dependence information

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Abstract

The evaluation of risks and risk bounds for joint portfolios is an important task in connection with the determination of risk capital as induced by regulatory prescriptions in finance and in insurance. It faces two basic problems. One is induced by the model risk arising from the use of precise but possibly incorrect models. On the other hand risk estimates based only on basic information as for example on the marginal (individual) risk distributions may be too wide to be usable in praxis. In this paper we survey some recent endeavor to include partial dependence and structural information in order to obtain reliable and usable improved risk bounds.

1 Introduction

For the evaluation of risks there are several structural and dependence models in use. The risk vector $X = (X_1, \dots, X_n)$ is described typically by specified marginal distributions and by some copula model describing the dependence structure. Alternatively there are several structural models like factor models in common use to describe the connection between the risks. Several basic statistical methods and techniques have been developed to construct estimators of the dependence structure like the empirical copula function (see Rüschendorf (1976), Deheuvels (1979), Stute (1984)) or the tail empirical copula as estimator for the (tail-) copula function. Similarly various estimators for dependence parameters as for the tail dependence index, for Spearmann's ϱ or for Kendall's τ have been introduced and used to test hypotheses on the dependence structure. (see f.e. Rüschendorf (1974), Genest, Ghoudi, and Rivest (1995), Genest, Rémillard, and Beaudoin (2009)). In many applications however there are not enough data available to use these methods in a reliable way. As a consequence there is a considerable amount of model risk when using these methods in an uncritical way. Many instances of these problems have been documented in the recent literature.

In recent years a lot of effort has been undertaken to base risk bounds only on reliable information available from the data, arising from history or from external sources. In particular the case where only information on the marginals is available while the dependence structure is completely unknown has been considered in detail starting with the paper of Embrechts and Puccetti (2006).

In Section 2 we give a brief review of this development. In the following sections we describe several recent approaches to introduce additional dependence information and structural information in order to tighten the risk bounds. In particular we consider higher order marginals, positive resp. negative dependence restrictions, independence information, variance and higher order moment bounds and partially specified risk factor models. The general insight obtained is that positive dependence information allows to increase lower risk bounds but typically not the upper risk bounds. Negative dependence information on the other hand allows to decrease upper risk bounds but typically not the lower risk bounds.

2 VaR- and TVaR bounds with marginal information

Let $X = (X_1, \dots, X_n)$ be a risk vector with marginals $X_i \sim F_i$, $1 \leq i \leq n$. Then the sharp tail risk bounds without dependence information are given by

$$M(s) = \sup_{X_i \sim F_i} P\left(\sum_{i=1}^n X_i \geq s\right) \quad \text{and} \quad m(s) = \inf_{X_i \sim F_i} P\left(\sum_{i=1}^n X_i \geq s\right). \tag{2.1}$$

Similarly, for the Value at Risk of the sum $S = \sum_{i=1}^n X_i = S_n$ we define the sharp VaR bounds

$$\overline{\mathrm{VaR}}_\alpha = \sup_{X_i \sim F_i} \mathrm{VaR}_\alpha(S) \quad \text{and} \quad \underline{\mathrm{VaR}}_\alpha = \inf_{X_i \sim F_i} \mathrm{VaR}_\alpha(S). \tag{2.2}$$

The dependence uncertainty (DU-)interval is defined as the interval $[\,\underline{\mathrm{VaR}}_\alpha, \overline{\mathrm{VaR}}_\alpha\,]$. Dual representations of (sharp) upper and lower bounds were given in Embrechts and Puccetti (2006) and in Puccetti and Rüschendorf (2012a). In some homogeneous cases exact sharp bounds were derived in Wang and Wang (2011) and extended in Puccetti and Rüschendorf (2013) resp. Puccetti, Wang, and Wang (2013) and in Wang (2014). Since the dual bounds are difficult to calculate in higher dimensions in the inhomogeneous case the development of the rearrangement algorithm (RA) in Puccetti and Rüschendorf (2012a) and extended in Embrechts, Puccetti, and Rüschendorf (2013) was an important step to approximate the sharp VaR bounds in a reliable way also in high dimensional examples.

As a result it has been found that the DU-interval typically is very wide. The comonotonic sum $S^c = \sum_{i=1}^n X_i^c$ is typically not the worst dependence structure and often the worst case VaR exceeds the comonotonic VaR denoted as VaR^+ by a factor of 2 or more as shown f.e. in the following two examples (see Table 2.1 and Figure 2.1). A detailed discussion of these effects is given in Embrechts et al. (2013).

α	$\underline{\mathrm{VaR}}_\alpha$ (RA range)	VaR^+_α (exact)	$\overline{\mathrm{VaR}}_\alpha$ (exact)	$\overline{\mathrm{VaR}}_\alpha$ (RA range)
0.99	530.12−530.24	5832.00	12302.00	12269.74−12354.00
0.995	562.33−562.50	8516.10	17666.06	17620.45−17739.60
0.999	608.08−608.47	19843.56	40303.48	40201.48−40467.92

Table 2.1 VaR-bounds, $n = 648$, $F_i = \text{Pareto}(2)$, $1 \le i \le n$

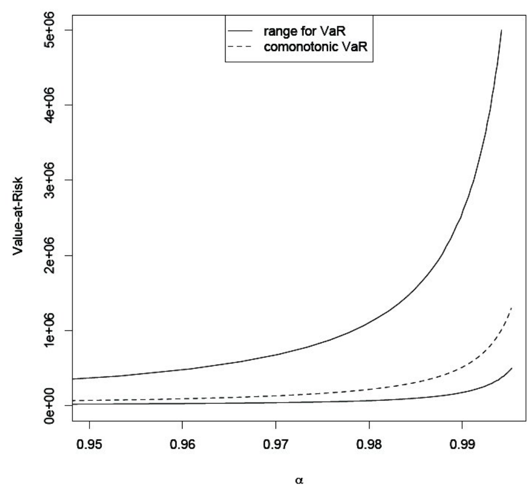

Figure 2.1 VaR-bounds, $d = 8$, GPD-risk for operational risk data from Moscadelli (2004)

The following theorem gives simple to calculate *unconstrained* bounds for the VaR in terms of the TVaR resp. the LTVaR defined as

$$\mathrm{TVaR}_\alpha(X) = \frac{1}{\alpha}\int_{1-\alpha}^{1} \mathrm{VaR}_u(X)du \quad \text{resp.} \quad \mathrm{LTVaR}_\alpha(X) = \frac{1}{\alpha}\int_0^{\alpha} \mathrm{VaR}_u(X)du. \tag{2.3}$$

Theorem 2.1 (unconstrained bounds)

$$\begin{aligned} A := \sum_{i=1}^n \mathrm{LTVaR}_\alpha(X_i) &= \mathrm{LTVaR}_\alpha(S_n^c) \\ &\le \mathrm{VaR}_\alpha(S_n) \le \mathrm{TVaR}_\alpha(S_n) \\ &\le \mathrm{TVaR}_\alpha(S_n^c) = \sum_{i=1}^n \mathrm{TVaR}_\alpha(X_i) =: B \end{aligned}$$

For these bounds see Wang and Wang (2011), Puccetti and Rüschendorf (2012a) and Bernard, Rüschendorf, and Vanduffel (2015a).

Puccetti and Rüschendorf (2014) found the astonishing result, that the sharp VaR bounds are asymptotically equivalent to the unconstrained TVaR bounds in Theorem 2.1 (in the homogeneous case under some regularity conditions) i.e.

$$\overline{\mathrm{VaR}}_\alpha \sim \mathrm{TVaR}_\alpha(S_n^c) \quad \text{and} \quad \underline{\mathrm{VaR}}_\alpha \sim \mathrm{LTVaR}_\alpha(S_n^c) \ \text{ as } n \to \infty. \tag{2.4}$$

This result was then extended to the inhomogeneous case in Puccetti et al. (2013) and Wang (2014). The worst case dependence structure has negative dependence in the upper part of the distribution. Construction of this mixing (negatively dependent) part is an interesting task in itself. As a result one obtains tools to determine VaR bounds also for the high dimensional and for the general inhomogeneous case based on marginal informations only. The bounds however are typically too wide to be applicable in praxis. As consequence it is necessary to include further information on the dependence structure in order to obtain tighter risk bounds.

3 Higher dimensional marginals

The class of all possible dependence structures can be restricted if some higher dimensional marginals are known. Let $\mathcal{E}$ be a system of subsets J of $\{1, \dots, n\}$ and assume that for $J \in \mathcal{E}$ $F_{X_J} = F_J$ is known. The class

$$\mathcal{F}_{\mathcal{E}} = \mathcal{F}(F_J; J \in \mathcal{E}) \subset \mathcal{F}(F_1, \dots, F_n) \tag{3.1}$$

resp. the corresponding class of distributions $M_{\mathcal{E}}$ is called generalized Fréchet class. In some applications f.e. some two-dimensional marginals additionally to the one-dimensional marginals might be known. The relevant tail risk bounds then are given by

$$M_{\mathcal{E}}(s) = \sup \{P(S \geq s); F_X \in \mathcal{F}_{\mathcal{C}}\} \ \text{ and } m_{\mathcal{E}}(s) = \inf \{P(S \geq s); F_X \in \mathcal{F}_{\mathcal{C}}\}. \tag{3.2}$$

Under some conditions a duality result corresponding to the simple marginal case has been established under the assumption $M_{\mathcal{E}} \neq \emptyset$ for various classes of functions φ as e.g. upper semicontinuous functions (see Rüschendorf (1984, 1991a), Kellerer (1988)). The duality theorem then takes the form:

$$M_{\mathcal{E}}(\varphi) = \sup \left\{ \int \varphi dP; P \in M_{\mathcal{E}} \right\} = \inf \left\{ \sum_{J \in \mathcal{E}} \int f_J dP_J; \sum_{J \in \mathcal{E}} f_J \circ \pi_J \geq \varphi \right\}. \tag{3.3}$$

The dual problem is however not easy to determine. For specific classes of indicator functions one can use the duality result to connect up with Bonferroni type bounds.

Let $(E_i, \mathcal{A}_i)$, $1 \leq i \leq n$ be measurable spaces and let for $J \in \mathcal{E}$, $P_J \in M^1(E_J, \mathcal{A}_J)$, $(E_J, \mathcal{A}_J) = \bigotimes_{j \in J} (E_j, \mathcal{A}_j)$, be a marginal system. The following class of improved Fréchet bounds was given in Rüschendorf (1991a).

Proposition 3.1 (Bonferroni type bounds) *Let $(E_i, \mathcal{A}_i)$, $1 \le i \le n$,$(P_J, J \in \mathcal{E})$ be a marginal system. For $A_i \in \mathcal{A}_i$ and $A_J = \prod_{j\in J} A_j$ the following estimates hold:*

1. $M_{\mathcal{E}}(A_1 \times \cdots \times A_n) \le \min_{J\in\mathcal{E}} P_J(A_J)$
2. *In the case that $\mathcal{E} = J_2^n = \{(i,j); i,j \le n\}$, and with $q_i = P_i(A_i^c)$, $q_{ij} = P_{ij}(A_i^c \times A_j^c)$ it holds:*

$$M_{\mathcal{E}}(A_1 \times \cdots \times A_n) \le 1 - \sum q_i + \sum_{i<j} q_{ij} \tag{3.4}$$

$$m_{\mathcal{E}}(A_1 \times \cdots \times A_n) \ge 1 - \sum q_i + \sup_{\tau\in T} \sum_{(i,j)\in\tau} q_{ij}, \tag{3.5}$$

where T is the class of all spanning trees of G_n, the complete graph of $\{1,\dots,n\}$.

Part 1. yields improved Fréchet bounds compared to the usual Fréchet bounds with marginal information only. Part 2. relates Fréchet bounds to Bonferroni bounds of higher order, and implies in particular improved bounds for the distribution function.

For particular cases of decomposable systems also conditional bounds were given in Rüschendorf (1991a) and applied to risk bounds in Embrechts et al. (2013). For non-overlapping systems $\mathcal{E} = \{J_1,\dots,J_m\}$ with $J_k \cap J_i = \emptyset$ for $i \ne k$ define $Y_r := \sum_{i\in J_r} X_i$, $H_r := F_{Y_r}$, $r = 1,\dots,m$ and $\mathcal{H} = \mathcal{F}(H_1,\dots,H_m)$. Then consider

$$\begin{aligned} M_{\mathcal{H}}(s) &= \sup\{P(Y_1 + \cdots + Y_m \ge s); F_Y \in \mathcal{H}\} \quad \text{and} \\ m_{\mathcal{H}}(s) &= \inf\{P(Y_1 + \cdots + Y_m \ge s); F_Y \in \mathcal{H}\} \end{aligned}$$

$M_{\mathcal{H}}$ and $m_{\mathcal{H}}$ are tail bounds corresponding to a simple marginal system with marginals H_i.

Proposition 3.2 (non-overlapping systems) *For a non-overlapping marginal system $\mathcal{E}$, holds:*

$$M_{\mathcal{E}}(s) = M_{\mathcal{H}}(s) \quad \text{and} \quad m_{\mathcal{E}}(s) = m_{\mathcal{H}}(s). \tag{3.6}$$

The following extension to general marginal systems was given in Embrechts and Puccetti (2010) and Puccetti and Rüschendorf (2012a). Let $\eta_i := \#\{J_r \in \mathcal{E}; i \in J_r\}$, $1 \le i \le n$. For a risk vector X with $F_X \in \mathcal{F}_{\mathcal{E}}$ define:

$$Y_r := \sum_{i\in J_r} \frac{X_i}{\eta_i}, \quad H_r := F_{Y_r}, \quad r = 1,\dots,m.$$

$\mathcal{H} = \mathcal{F}(H_1,\dots,H_m)$ denotes the corresponding Fréchet class.

Proposition 3.3 (reduced Fréchet bounds) *Let $\mathcal{F}_{\mathcal{E}} \neq \emptyset$ be a consistent marginal system, then for $s \in \mathbb{R}$ holds*

$$M_{\mathcal{E}}(s) \leq M_{\mathcal{H}}(s) \quad \text{and} \quad m_{\mathcal{E}}(s) \geq m_{\mathcal{H}}(s). \tag{3.7}$$

In comparison to the non-overlapping case the bounds in (3.7) are not sharp in general but they can be determined numerically. The RA algorithm can be used to calculate the reduced Fréchet bounds $M_{\mathcal{H}}$ and $m_{\mathcal{H}}$. In order to apply the reduced bounds in Proposition 3.2, 3.3 it is enough to know the partial sum distributions H_r instead of the multivariate marginal distributions F_{J_r}.

Also generalized weighting schemes of the form

$$Y_r^\alpha = \sum_{i=1}^m \alpha_i^r X_i, \text{ with } \alpha_i^r > 0 \text{ iff } i \in J_r \text{ and } \sum_{r=1}^m \alpha_i^r = 1$$

have been introduced, leading to a parametrized family of bounds.

The magnitude of reduction of the reduced bounds $\overline{\mathrm{VaR}}_\alpha^r$ compared to the unconstrained upper bound $\overline{\mathrm{VaR}}_\alpha$ and the comonotonic VaR^+ depends on the structure of the two-dimensional marginals. In the following example we assume that there are $n = 600$ Pareto(2) risks and that the two-dimensional marginals are comonotonic in case A) and independent in case B). The results confirm the intuition, that in case A) the improvement is moderate while in case B) it is considerable (see Figure 3.1, Table 3.1).

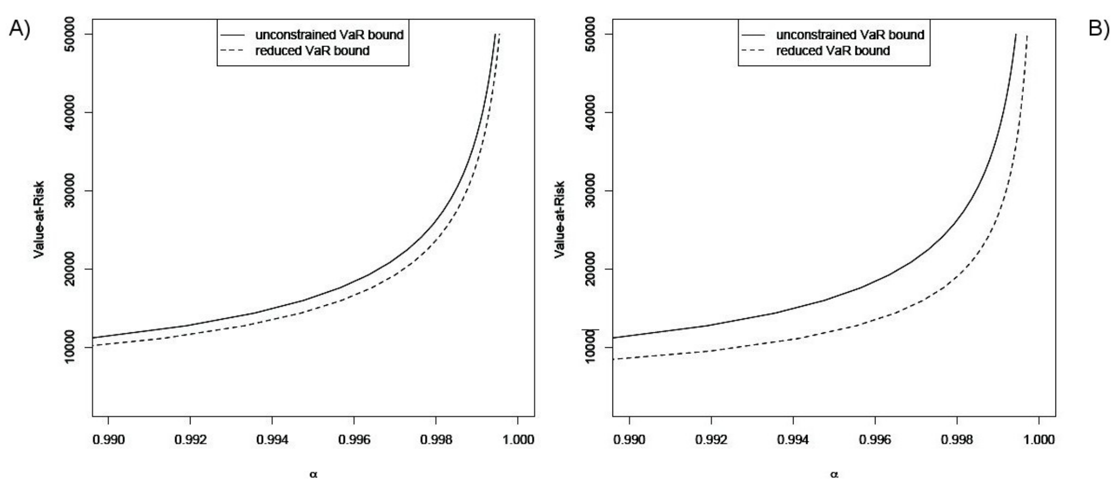

Figure 3.1 reduced bounds $n = 600$ Pareto(2) variables, A $\sim$ comonotone $F_{2j-1,2j}$ marginals, B $\sim$ independent $F_{2j-1,2j}$ marginals

α	VaR_α^+	$\overline{\mathrm{VaR}}_\alpha^{r,\mathrm{A}}$	$\overline{\mathrm{VaR}}_\alpha^{r,\mathrm{B}}$	$\overline{\mathrm{VaR}}_\alpha(L)$
0.99	5400.00	10309.14	8496.13	11390.00
0.995	7885.28	14788.71	12015.04	16356.42
0.999	18373.67	33710.3	26832.2	37315.70

Table 3.1 reduced bounds as in Figure 3.1

As a result it is found that higher order marginals may lead to a considerable reduction of VaR bounds, when the known higher dimensional marginals do not specify strong positive dependence. For various applications like in insurance applications however this kind of higher oder marginals information F_{J_r} or H_r may not be available.

4 Risk bounds with variance and higher order moment constraints

In several applications like in typical insurance applications it may be possible to have information available on bounds for the variance or for higher order moments of the portfolio. Consider therefore information of the form:

$$X_i \sim F_i,\ i \leq i \leq n \quad \text{and} \quad \mathrm{Var}(S_n) \leq s^2. \tag{4.1}$$

Alternatively also partial information on some of the covariances $\mathrm{Cov}(X_i, X_j)$ may be available. The corresponding optimization problems

$$\begin{aligned} M &= M(s^2) = \sup\{\mathrm{VaR}_\alpha(S_n);\ S_n \text{ satisfies (4.1)}\} \quad \text{and} \\ m &= m(s^2) = \inf\{\mathrm{VaR}_\alpha(S_n);\ S_n \text{ satisfies (4.1)}\} \end{aligned} \tag{4.2}$$

have been considered in Bernard et al. (2015a). A variant of the Cantelli bounds then is given as follows:

Theorem 4.1 (VaR bounds with variance information) *Let* $\alpha \in (0,1)$ *and* $\mathrm{Var}(S_n) \leq s^2$*, then*

$$\begin{aligned} a := \max\left(\mu - s\sqrt{\frac{\alpha}{1-\alpha}}, A\right) &\leq m \leq \mathrm{VaR}_\alpha(S_n) \\ &\leq M \leq b := \min\left(\mu + s\sqrt{\frac{\alpha}{1-\alpha}}, B\right) \quad \textit{where } \mu = \mathrm{E}S_n\,. \end{aligned} \tag{4.3}$$

The bounds in (4.3) are simple to evaluate and depend only on the variance bound s, on the mean μ as well as on the unconstrained bounds A, B.

The VaR bounds and the convex order worst case dependence structure depend on convex order minima in the upper and in the lower part $\{S_n \geq \mathrm{VaR}_\alpha(S_n)\}$

resp. $\{S_n < \mathrm{VaR}_\alpha(S_n)\}$ of the distribution of S_n. This is described in the following proposition (cf. Bernard et al. (2015a)). Let for $X_i \sim F_i$, $q_i(\alpha)$ denote the upper α-quantile of X.

Proposition 4.2 *Let $X_i \sim F_i$, $F_i^\alpha \sim F_i/[q_i(\alpha), \infty)$ and let X_i^α, $Y_i^\alpha \sim F_i^\alpha$, then:*

a) $M = \sup\limits_{X_i \sim F_i} \mathrm{VaR}_\alpha\left(\sum_{i=1}^n X_i\right) = \sup\limits_{Y_i^\alpha \sim F_i^\alpha} \mathrm{VaR}_0\left(\sum_{i=1}^n Y_i^\alpha\right)$

b) If $S^\alpha = \sum_{i=1}^n Y_i^\alpha \leq_{\mathrm{cx}} \sum_{i=1}^n X_i^\alpha$, *then*

$$\mathrm{VaR}_0\left(\sum_{i=1}^n X_i^\alpha\right) \leq \mathrm{VaR}_0(S^\alpha) = \operatorname{ess\,inf}\left(\sum_{i=1}^n Y_i^\alpha\right) \leq B$$

Thus maximizing of VaR corresponds to maximizing the minimal support over all $Y_i \sim F_i^\alpha$ and it is implied by convex order. This connection is intuitively explainable. An extreme dependence structure for the maximization is obtained when the random variables are mixable in the upper resp. the lower part of the distribution. In the following Figure 4.1 this is applied to the quantile function in the comonotonic case and leads to an increase of the upper resp. decrease of the lower value of VaR if the distribution of S_n is mixable on the upper resp. lower part of the distribution.

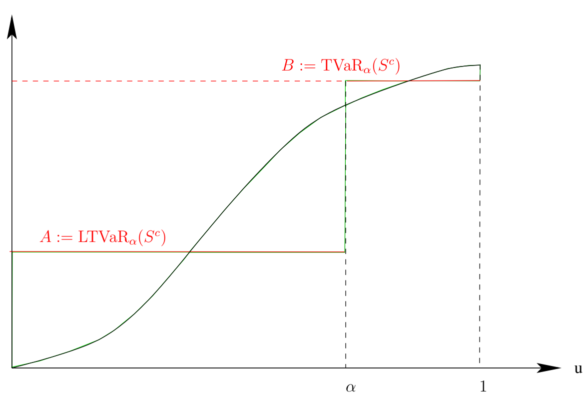

Figure 4.1 VaR bounds and convex order

The connection to the convex order gives the motivation for the extended rearrangement algorithm (ERA) a variant of the RA. This algorithm consists of two alternating steps:

1. choice of domain, starting from largest α-domain
2. rearrangement in the upper α-part and in the lower $1-\alpha$-part

3. check if the variance constraint is fulfilled
4. shift the domain and iterate

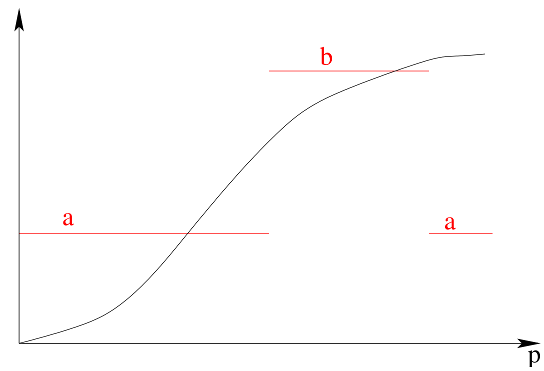

Figure 4.2 ERA algorithm

Also a variant of the algorithm has been introduced which uses self determined splits of the domain. The following Table 4.1 compares for a portfolio of $n = 100$ Pareto(3) distributed risks the approximate sharp bounds (m, M) calculated by the ERA for various variance restrictions, determined by constant pairwise correlations ϱ with the VaR bounds (a, b) and the unconstrained bounds (A, B).

We find considerable improvements over the unconstrained bounds (A, B) for small variance levels. Since the ERA bounds correspond to valid dependence structures and are close to the theoretical bounds (a, b) this shows that the bounds (a, b) are good and also that the ERA works well.

(m, M)	$\varrho = 0$	$\varrho = 0.15$	$\varrho = 0.3$
$\mathrm{VaR}_{0.95}$	(47.96; 84.72)	(42.48; 188.9)	(39.61; 243.3)
$\mathrm{VaR}_{0.99}$	(48.99; 129.5)	(46.61; 366.0)	(45.36; 489.5)
$\mathrm{VaR}_{0.995}$	(49.23; 162.8)	(47.54; 499.1)	(46.68; 671.5)

(a, b)	$\varrho = 0$	$\varrho = 0.15$	$\varrho = 0.3$	(A, B)	
$\mathrm{VaR}_{0.95}$	(47.96; 84.74)	(42.48; 188.9)	(39.61; 243.4)	$\mathrm{VaR}_{95\%}$	(36.46; 303.3)
$\mathrm{VaR}_{0.99}$	(48.99; 129.6)	(46.59; 367.3)	(45.33; 491.7)	$\mathrm{VaR}_{99\%}$	(44.47; 577.6)
$\mathrm{VaR}_{0.995}$	(49.23; 162.9)	(47.54; 500.0)	(46.65; 676.3)	$\mathrm{VaR}_{99.5\%}$	(46.33; 741.1)

Table 4.1 VaR bounds and ERA with unconstrained bounds for Pareto(3) variables, $n = 100$

In an application to a credit risk portfolio of $n = 10000$ binomial loans $X_j \sim \mathcal{B}(1, p)$ with default probability $p = 0.049$ and variance $s^2 = np(1-p) + n(n-1)p(1-p)\varrho^D$ where the default correlation is $\varrho^D = 0.0157$, Bernard et al. (2015a) compared the unconstrained and constrained bounds with some standard industry models like

KMV, Beta and Credit Metrics. The following table shows the improvement of the variance constrained bounds and also the still considerable dependence uncertainty. It raises some doubts on the reliability of the standard models used in practice.

	(A, B)	(a, b)	(m, M)	KMV	Beta	Credit Metrics
$\mathrm{VaR}_{0.8}$	(0%; 24.50%)	(3.54%; 10.33%)	(3.63%; 10%)	6.84%	6.95%	6.71%
$\mathrm{VaR}_{0.9}$	(0%; 49.00%)	(4.00%; 13.04%)	(4.00%; 13%)	8.51%	8.54%	8.41%
$\mathrm{VaR}_{0.95}$	(0%; 98.00%)	(4.28%; 16.73%)	(4.32%; 16%)	10.10%	10.01%	10.11%

Table 4.2 VaR bounds compared to some standard models (KMV, Beta, Credit Metrics)

It is found that the amount of reduction of the VaR bounds can be considerable when the variance bound s^2 is small enough. Additional higher order moment restrictions of the form $ES_n^k \leq c_k$, $2 \leq k \leq K$ are considered in Bernard, Denuit, and Vanduffel (2014), Bernard, Rüschendorf, Vanduffel, and Yao (2015b) . The following table shows the potential of higher order moments in a specific case for a corporate portfolio.

VaR assessment of a corporate portfolio

	$q =$	KMV	Comon.	Unconstrained	$K = 2$	$K = 3$	$K = 4$
	0.95	340.6	393.3	$(34.0; 2083.3)$	$(97.3; 614.8)$	$(100.9; 562.8)$	$(100.9; 560.6)$
$\varrho = 0.10$	0.99	539.4	2374.1	$(56.5; 6973.1)$	$(111.8; 1245.0)$	$(115.0; 941.2)$	$(115.9; 834.7)$
	0.995	631.5	5088.5	$(89.4; 10119.9)$	$(114.9; 1709.4)$	$(117.6; 1177.8)$	$(118.5; 989.5)$

Table 4.3 VaR bounds with higher order moment constraints $\varrho = 0.10$, $n = 100$, models as in Table 4.2

The variance resp. moment restriction is a global negative dependence assumption. Therefore one can expect from this assumption a reduction of the upper VaR bounds as shown in the examples. The effect on an improvement of lower bounds is of minor magnitude.

5 Dependence / Independence information

How does positive, negative or independence information influence risk bounds? A weak notion of positive dependence is the positive orthant dependence (POD). X is called *positive upper orthant dependent* (PUOD) if

$$\overline{F}_X(x) = P(X \geq x) \geq \prod_{i=1}^n P(X_i \geq x_i) = \prod_{i=1}^n \overline{F}_i(x_i).$$

X is called *positive lower orthant dependent* (PLOD) if

$$F_X(x) \geq \prod_{i=1}^n F_i(x_i), \ \forall x.$$

X is POD if X is PLOD and PUOD.

More generally for $F = F_X$, $\overline{F} = \overline{F}_X$ let G be an increasing function with $F^- \leq G \leq F^+$; F^-, F^+ the Fréchet bounds and let H be a decreasing function with $\overline{F}^- \leq H \leq \overline{F}^+$. Further let $\leq_{\text{uo}}$, $\leq_{\text{lo}}$ denote the upper resp. lower orthant ordering. Then

$G \leq F$ is a *positive dependence restriction* on the lower tail probabilities and

$H \leq \overline{F}$ is a *positive dependence restriction* on the upper tail probabilities.

In the case that G is a distribution function and H is a survival function these conditions correspond to ordering conditions w.r.t. $\leq_{\text{lo}}$ resp. $\leq_{\text{uo}}$. In the case that $G(x) = \prod F_i(x_i)$, these conditions together are equivalent to X being POD.

Similarly: $F \leq H$, $\overline{F} \leq H$ are *negative dependence restrictions.*

These kind of restrictions have been discussed in a series of papers, as in Williamson and Downs (1990), Denuit, Genest, and Marceau (1999), Denuit, Dhaene, and Ribas (2001), Embrechts, Höing, and Juri (2003), Rüschendorf (2005), Embrechts and Puccetti (2006), Puccetti and Rüschendorf (2012a). As a result the following improved standard bounds are obtained (see Puccetti and Rüschendorf (2012a)).

Theorem 5.1 (positive dependence restriction, improved standard bounds) *Let X be a risk vector with marginals $X_i \sim F_i$. Let G be an increasing function with $F^- \leq G \leq F^+$ and let H be a decreasing function with $\overline{F}^- \leq H \leq \overline{F}^+$. Then*

a) If $G \leq F_X$, then

$$P\left(\sum_{i=1}^{d} X_i \leq s\right) \geq \bigvee G(s); \tag{5.1}$$

b) If $H \leq \overline{F}_X$, then

$$P\left(\sum_{i=1}^{d} X_i < s\right) \leq 1 - \bigvee H(s); \tag{5.2}$$

c) If F is POD, then

$$\begin{aligned} \bigvee\left(\prod_{i=1}^{d} F_i\right)(s) &\leq P\left(\sum_{i=1}^{d} X_i \leq s\right), \\ P\left(\sum_{i=1}^{d} X_i < s\right) &\leq 1 - \bigvee\left(\prod_{i=1}^{d} \overline{F}_i\right)(s), \end{aligned} \tag{5.3}$$

where with $U(s) := \left\{x \in \mathbb{R}^n; \sum_{i=1}^{n} x_i = s\right\}$, $\bigwedge G(s) := \inf_{x \in U(s)} G(x)$ is the G-infimal convolution, $\bigvee H(s) := \sup_{x \in U(s)} H(x)$ is the G-supremal convolution.

Bignozzi, Puccetti, and Rüschendorf (2015) considered the following specific type of model assumption to explore the consequences of this kind of dependence assumptions. Let the risk vector $X = (X_1, \ldots, X_n)$ have marginals $F_i = F_{X_i}$ and assume that $\{1, \ldots, n\} = \bigcup_{j=1}^{k} I_j$ is a split into k subgroups. Let $Y = (Y_1, \ldots, Y_n)$ be a random vector, that satisfies

$$F_Y(x) = \prod_{j=1}^{k} \min_{i \in I_j} G_j(x_i), \tag{5.4}$$

i.e. Y has k independent homogeneous subgroups and the components within the subgroup I_j are comonotonic. The basic assumption made is

$$Y \le X \tag{5.5}$$

where $\le$ is the upper or lower positive orthant ordering $\le_{\text{uo}}$ or $\le_{\text{lo}}$.

In case $F_i = G_j$ for $i \in I_j$ and $k = n$, (5.5) is equivalent to X being PUOD resp. PLOD. As k decreases the assumption is getting stronger and for $k = 1$ it amounts to the strictest assumption that X is comonotonic. In Bignozzi et al. (2015) an analytic expression for the upper and lower bounds $\mathrm{VaR}_\alpha^{\text{ub}}$, $\mathrm{VaR}_\alpha^{\text{lb}}$ under this assumption is given. It turns out that as expected the upper VaR bounds are only slightly improved. The lower bounds are improved strongly if k is relatively small. For $k = n$ there is no improvement of the unconstrained lower VaR bounds $\underline{\mathrm{VaR}}_\alpha$. The POD assumption alone is too weak to lead to improved lower bounds (see Table 5.1).

$n = 8$	$k = 1$		$k = 2$	$k = 4$	$k = 8$
α	$\underline{\mathrm{VaR}}_\alpha$	$\mathrm{VaR}_\alpha^{\text{lb}}$	$\mathrm{VaR}_\alpha^{\text{lb}}$	$\mathrm{VaR}_\alpha^{\text{lb}}$	$\mathrm{VaR}_\alpha^{\text{lb}}$
0.990	9.00	72.00	36.00	18.00	9.00
0.995	13.14	105.14	52.57	26.28	13.14

Table 5.1 n homogeneous Pareto(2) risks, split into $\frac{n}{k}$ subgroups of equal size

Similar conclusions are also obtained for inhomogeneous cases.

A stronger notion of positive dependence is the (sequential) positive cumulative dependence (PCD) defined by

$$P\left(\sum_{i=1}^{k-1} X_i > t_1 \mid X_k > t_2\right) \ge P\left(\sum_{i=1}^{k-1} X_i > t_1\right), \quad 2 \le k \le n \tag{5.6}$$

This is a sequential version of the PCD notion in Denuit et al. (2001). Similarly, (sequential) negative cumulative dependence (NCD) is defined if "$\le$" holds in (5.6).

From the PCD assumption one obtains the following result

Proposition 5.2 *Let* $S_n^\perp = \sum_{i=1}^n X_i^\perp$ *denote the independent sum with* $X_i^\perp \sim F_i$.

a) If X *is PCD, then* $S_n^\perp \le_{\mathrm{cx}} S_n$

b) If X *is NCD, then* $S_n \le_{\mathrm{cx}} S_n^\perp$

This result implies as consequence the following VaR resp. TVaR bounds.

Corollary 5.3 (positive dependence restriction) *If* X *is PCD, then*

a) $\mathrm{TVaR}_\alpha(S_n^\perp) \le \mathrm{TVaR}_\alpha(S_n)$

b) $\mathrm{LTVaR}_\alpha(S_n^\perp) \le \mathrm{LTVaR}_\alpha(S_n) \le \mathrm{VaR}_\alpha(S_n) \le \mathrm{TVaR}_\alpha(S_n^c)$

The stronger PCD notion implies improvements of the lower bounds for VaR and for TVaR. Under the corresponding negative dependence assumption one obtains improvements of the upper bounds.

Proposition 5.4 (negative dependence restriction) *If* X *is NCD, then*

a) $S_n \le_{\mathrm{cx}} S_n^\perp$ *and*

b) $\mathrm{VaR}_\alpha(S_n) \le \mathrm{TVaR}_\alpha(S_n) \le \mathrm{TVaR}_\alpha(S_n^\perp)$

Remark 5.5 *A stronger positive dependence ordering between any two random vectors* X *and* Y*, the WCS = the weakly conditionally in sequence ordering was introduced in Rüschendorf (2004).*

$$X \le_{\mathrm{wcs}} Y \text{ implies that } \sum_{i=1}^n X_i \le_{\mathrm{cx}} \sum_{i=1}^n Y_i. \tag{5.7}$$

This ordering notion allows to pose more general kinds of positive (negative) dependence restrictions and to compare not only to the independent case. Several examples for applications of this ordering are given in that paper.

In the subgroup example the WCS condition is strong enough to imply strongly improved lower bounds for $k \le n$ subgroups also in the case that $k = n$ (see Table 5.2).

$n=8$	unconstrained		$k=1$	$k=2$	$k=4$	$k=8$
α	$\underline{\mathrm{ES}}_\alpha$	$\overline{\mathrm{ES}}_\alpha$	$\mathrm{ES}_\alpha^{\mathrm{lb}}$	$\mathrm{ES}_\alpha^{\mathrm{lb}}$	$\mathrm{ES}_\alpha^{\mathrm{lb}}$	$\mathrm{ES}_\alpha^{\mathrm{lb}}$
0.990	12.00	38.27	38.27	29.15	23.29	19.56
0.995	12.00	41.64	41.64	31.15	24.52	20.33

Table 5.2 $n = 8$, Gamma distributed risk, 4 Gamma (2, 1/2), 4 Gamma (4, 1/2)

The reduction of the DU-spread in this example ranges from about 28 % for $k = 8$ to 65 % for $k = 2$.

A particular relevant case of reduction of the VaR bounds arises under the independence assumption I) which was discussed in Puccetti, Rüschendorf, Small, and Vanduffel (2015).

I) The subgroups $I_1, \dots, I_k$ are independent.

In this case we can represent the sum S as an independent sum

$$S = \sum_{i=1}^{k} Y_i \quad \text{where} \quad Y_i = \sum_{j \in I_i} X_j. \tag{5.8}$$

We denote by $S^{c,k} = \sum_{i=1}^{k} Y_i^c$ the comonotonic version of the sum.

Theorem 5.6 *Under the independence assumption* I) *holds:*

$$\begin{aligned} a^{\mathrm{I}} := \mathrm{LTVaR}_\alpha(S^{c,k}) \le \underline{\mathrm{VaR}}_\alpha^{\mathrm{I}} \le \overline{\mathrm{VaR}}_\alpha^{\mathrm{I}} \\ \le b^{\mathrm{I}} := \mathrm{TVaR}_\alpha(S^{c,k}). \end{aligned}$$

Note that the upper and lower bounds a^{I}, b^{I} can be calculated numerically by Monte Carlo simulation. As consequence one obtains strongly improved VaR bounds $a^{\mathrm{I}}, b^{\mathrm{I}}$ compared to the sharp VaR bounds as is demonstrated for a Pareto example in Table 5.3.

$(a^{\mathrm{I}}, b^{\mathrm{I}})$	$k = 1$	$k = 2$	$k = 5$	$k = 25$	$k = 50$	$(\underline{\mathrm{VaR}}_\alpha; \overline{\mathrm{VaR}}_\alpha)$
$\alpha = 0.990$	(18.23; 153.72)	(20.21; 116.32)	(22.03; 81.54)	(23.76; 48.57)	(24.15; 41.09)	(18.24; 153.3)
$\alpha = 0.995$	(22.24; 297.84)	(23.14; 208.2)	(23.92; 132.28)	(24.59; 65.87)	(24.73; 51.98)	(22.26; 297.64)

Table 5.3 $n = 50$, Pareto(3) variables

The bounds in Theorem 5.6 have also been extended to the case of partial independent substructures which appear to be realistic models in several important applications like in hierarchical insurance models (containing several independencies). It has been applied to a real insurance example in dimension $n = 11$ and with $k = 4$ independent subgroups.

Let $I_1, \dots, I_4$ be risks which are modeled in the insurance company $I_1 = \{$market-, credit-, insurance-, business-, asset-, non life-, reput.-, and life risk$\}$ by Gaussian marginals. Further denote by $I_2 = \{$reinsurance risk$\}$, $I_3 = \{$operational risk$\}$ risks which are modeled by log-Normal distributions and finally let $I_4 = \{$catastrophic risk$\}$ be a risk modeled by a Pareto distribution. The independence assumption leads to a considerable reduction of approximatively 30 % of the upper risk bound (see Table 5.4) which is even a strong improvement over the comonotonic case.

α	b^{I}	VaR_α^+	$\overline{\mathrm{VaR}}_\alpha$
0.990	147.34 − 149.66	168.37	209.59
0.995	173.37 − 176.96	202.89	249.55
0.999	250.41 − 262.47	304.63	367.70

Table 5.4 comparison of b^{I}, VaR_α^+, and $\overline{\mathrm{VaR}}_\alpha$ for a insurance portfolio, $n = 11$

An analysis shows that in this example the independence information is dominating the variance information, i.e. the independence bounds improve on the variance based bounds. The results in this example yield upper risk bounds which are based on reliable information and are acceptable for the application considered.

6 Partially specified risk factor models

In Bernard, Rüschendorf, Vanduffel, and Wang (2016) risk bounds are discussed under additional structural information. It is assumed that the risk vector is described by a

$$\textbf{factor model:} \quad X_j = f_j(Z, \varepsilon_j), \quad 1 \le j \le n \tag{6.1}$$

where Z is a systemic risk factor and ε_j are individual risk factors. It is assumed that the joint distributions H_j of (X_j, Z) are known $1 \le j \le n$, but the joint distribution of (ε_j) and Z is not specified as is done in the usual factor models. Therefore, this describes partially specified risk factors models without the usual assumptions of conditional independence of (ε_j) given the risk factor Z.

In particular the marginal distributions $F_{j|z}$ of X_j given $Z = z$ are known. The set of admissible models consistent with this partial specification is denoted by $A(H)$ where $H = (H_j)$. The idea underlying this approach is that the common risk factor Z should reduce the DU-interval. This model assumption reduces the upper VaR bounds $\overline{\mathrm{VaR}}_\alpha^f$ over the class of admissible models if Z generates negative dependence and it increases the lower VaR bounds $\underline{\mathrm{VaR}}_\alpha^f$ when Z induces positive dependence.

The partially specified factor model can be described by a mixture representation $X = X_Z$ with $X_z = (X_{j,z}) \in A(F_z)$, $F_z = (F_{j|z})$, where Z and $(X_{j,z})$ are independent. Then

$$F_S = \int F_{S_z}\, dG(z) \quad \text{with } G \sim Z. \tag{6.2}$$

Let $q_z(\alpha) = \mathrm{VaR}_\alpha(S_z)$ denote the VaR of S_z at level α and define for $\gamma \in \mathbb{R}^1$, $\gamma_z = q_z^{-1}(\gamma)$ the inverse γ-quantile of S_z i.e. the amount of probability chosen from $\{Z = z\}$. Further define

$$\gamma^*(\beta) = \inf\left\{\gamma \in \mathbb{R};\, \int \gamma_z\, dG(z) \ge \beta\right\}. \tag{6.3}$$

From the mixture representation in (6.1) the following mixture representation of $\mathrm{VaR}_\alpha(S_Z)$ and of the worst case $\overline{\mathrm{VaR}}_\alpha^f$ w.r.t. the admissible class is derived.

Theorem 6.1 (worst case VaR in partially specified factor model) *For $\alpha \in (0,1)$ holds:*

a) $\mathrm{VaR}_\alpha(S_Z) = \gamma^*(\alpha)$

b) $$\overline{\mathrm{VaR}}_\alpha^f = \overline{\gamma}^*(\alpha) = \inf\left\{\gamma; \int \overline{\gamma}_z dG(z) \geq \alpha\right\}, \tag{6.4}$$

where $\overline{q}_z(\alpha) = \overline{\mathrm{VaR}}_\alpha(S_z)$, $\overline{\gamma}_z = (\overline{q}_z)^{-1}(\gamma)$ is the worst case inverse γ-quantile.

The mixture representation in (6.4) has an obvious intuitive meaning. It is however in general not simple to calculate. For that purpose it is useful to replace the conditional VaR's in formula (6.4) by conditional TVaR's which are easy to calculate, i.e. define

$$t_z(\beta) = \mathrm{TVaR}_\beta(S_z^c) = \sum_{j=1}^n \mathrm{TVaR}_\beta(X_{j,z}). \tag{6.5}$$

Then $q_z(\beta) \leq t_z(\beta)$ and we obtain

$$\overline{\gamma}^*(\beta) \leq \gamma_t^*(\beta) = \inf\left\{\gamma; \int t_z^{-1}(\gamma) dG(z) \geq \beta\right\}. \tag{6.6}$$

As a result this estimate from above leads to the following corollary.

Corollary 6.2 (TVaR bounds for the partially specified risk factor model)

a) $$\overline{\mathrm{VaR}}_\alpha^f = \overline{\gamma}^*(\alpha) \leq \gamma_t^*(\alpha). \tag{6.7}$$

b) With $T_z^+ := \mathrm{TVaR}_U(S_z^c)$, $U \sim U(0,1)$, the following representation holds

$$\mathrm{VaR}_\alpha(T_Z^+) = \gamma_t^*(\alpha). \tag{6.8}$$

The expression in (6.8) is well suited for Monte Carlo simulations and thus for the numerical calculation of upper bounds for $\overline{\mathrm{VaR}}_\alpha^f$. The following example confirms the idea of the influence of the systemic risk factor Z on the reduction of the DU-spread.

Example 6.3 *Consider the case $n = 2$ where*

$$X_1 = (1-Z)^{-1/3} - 1 + \varepsilon_1$$
$$X_2 = p\left((1-Z)^{-1/3} - 1\right) + (1-p)\left(Z^{-1/3} - 1\right) + \varepsilon_2$$

where $Z \sim U(0,1)$, $\varepsilon_i \sim$ Pareto(4) and $p \in [0,1]$ is a dependence parameter. For small p the common risk factor produces strong negative dependence, for large p it produces strong positive dependence. Therefore, for $p \approx 0$ we expect a strong reduction of the upper risk bounds; for $p \approx 1$ we expect a strong improvement of the lower risk bound. This is confirmed in Figure 6.1 for the case $\alpha = 0.90$.

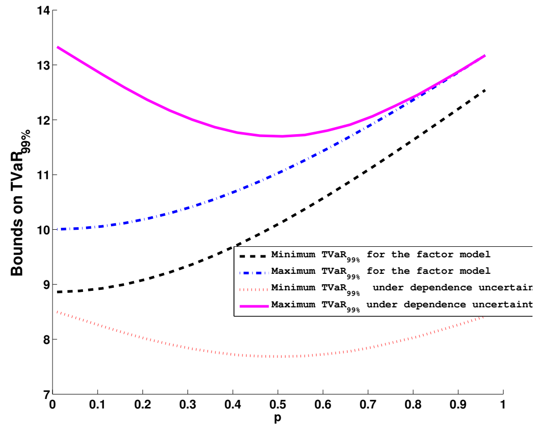

Figure 6.1 TVaR reduction in factor model independence of DU-spread on p

Similar reduction results are also obtained at other confidence levels α for VaR and hold true also in higher dimensional examples (see Bernard et al. (2016)). For strong negative dependence we see a strong reduction of the upper bounds, for strong positive dependence induced by the common risk factor Z we obtain a strong improvement of the lower bound. But for all possible values of the dependence parameter p the reduction of the DU-spread is of similar order. In our example above it is of order of 60 - 70 % which is due to the dominant influence of the common risk factor Z.

The consideration of partially specified risk factor models is a flexible and effective tool to reduce DU-spreads. The magnitude of the reduction amounts to the influence of the common risk factor Z. Examples of particular interest for applications are the *Bernoulli mixture* models for credit risk portfolios where the conditional distributions $F_{i|z}$ of X_i given $Z = z$ are given by $B(1, p_i(z))$. Common models for financial portfolios are the multivariate normal mean-variance mixture models of the form

$$X_i = \mu_i + \gamma_i Z + \sqrt{Z} \varrho_i \varepsilon_i, \quad 1 \le i \le n \tag{6.9}$$

where Z is a stochastic factor and ε_i are standard normal distributed. These models include many of the standard and well established marginal distributions in finance like Variance Gamma, hyperbolic or Normal Inverse Gaussian distributions. In our partially specified factor model we dismiss with the usual Gaussian dependence among the ε_i.

The results on partially specified risk factor models described above can be extended to more general *mixture models*. Let $D = D_1 + D_2 + D_3$ be a decomposition of the state space D of Z. Assume that for states $z \in D_1$ of the risk factor Z we have available a precise model P_z^1 for the risk vector X given $Z = z$ while for states $z \in D_2$ we have available the conditional distributions $F_z = (F_{j|z})$ i.e. the partially specified distributions. For $z \in D_3$ we only have available marginal information

(G_j). As result we obtain a mixture model of the form

$$P^X = \int_{D_1} P_z^1 \, dP^Z(z) + \int_{D_2} P_z^2 \, dP^Z(z) + p_3 P^3 \tag{6.10}$$

with P_z^1 completely specified for $z \in D_1$, $P_z^2 \in A(F_z)$ for $z \in D_2$ and $P^3 \in A(G_j)$. With $p_i = P(Z \in D_i)$ the model in (6.10) has three components

$$P^X = p_1 P^1 + p_2 P^2 + p_3 P^3, \tag{6.11}$$

where the (normalized) first component P^1 is explicitly modeled, the second one P^2 contains partially specified risk factor information, and the third one P^3 contains only marginal information.

Since

$$P\left(\sum_{j=1}^n X_j \geq t\right) = \sum_{i=1}^3 p_i P^i \left(\sum_{j=1}^n X_j \geq t\right) \tag{6.12}$$

we obtain the sharp tail risk bound for this extended mixture model

$$\overline{M}(t) = p_1 P^1 \left(\sum_{j=1}^n X_j \geq t\right) + p_2 \int_{D_2} \overline{M}_{2,z}(t) dP^Z(z) + p_3 \overline{M}_3(t), \tag{6.13}$$

where $\overline{M}_{2,z}(t)$ is the constrained tail risk bound in D_2 and $\overline{M}_3(t)$ is the marginal tail risk bound in D_3. The convex sharp upper bound in this model is given by

$$S = \sum_{i=1}^n X_i \leq_{\mathrm{cx}} I(Z \in D_1) F_1^{-1}(U) + I(Z \in D_2) S_{2,Z}^c + I(Z \in D_3) S_3^c, \tag{6.14}$$

where F_1 is the distribution function of $\sum_{j=1}^n X_i$ under P^1, $S_{2,z}^c = \sum_{j=1}^n F_{j|z}^{-1}(U)$ and $S_3^c = \sum_{j=1}^n G_j^{-1}(U)$ are the conditional resp. unconditional comonotonic vectors, $U \sim U(0,1)$ independent of Z. The formula in (6.14) implies directly sharp upper bounds for the Tail Value at Risk of S.

Also the TVaR upper bounds in Corollary 6.2 generalize to this extended mixture model since they are based only on the convex ordering properties as in (6.14).

An interesting case of this general model is the case where $D = \{0, 1\}$ and where for $z = 0$ we have an exact model in the central part of the distribution in $\mathbb{R}^n$ and for $z = 1$ we have only marginal information. The model has been suggested and analyzed in Bernard and Vanduffel (2015). In particular, the reduction of tail risk of the distribution of S for moderate levels α by the exactly modeled central part of the distribution is of practical relevance.

7 Conclusion

Sharp risk bounds for portfolios where only marginal information is available can be calculated by the RA-algorithm. They are however typically to wide to be usable in applications. Therefore, various further reductions of the VaR bounds have been proposed in the literature and are discussed in this paper. These are based on additional dependence or structural information.

Higher order marginals may give a good reduction of th DU-bounds when available. Variance constraints and also higher order moment constraints are often available and yield a good reduction when the constraints are small enough.

Partial dependence information together with structural information on subgroups can lead to interesting improvements, when the dependence notion used is strong enough. The weak positive orthant dependence (POD) alone is not sufficient. Of particular interest for applications is to include some (structural) independence information on the underlying model.

A particular flexible method to introduce relevant structural information is based on partially specified risk factor models. These models can be used based on realistic model information and often give a considerable improvement of the DU-spread depending on the magnitude of the influence of the common risk factor. We also briefly describe in this paper an extension of this approach to a more general class of mixture models.

References

C. Bernard and S. Vanduffel. A new approach to assessing model risk in high dimensions. *Journal of Banking and Finance*, 58:166–178, 2015. doi: 10.1016/j.jbankfin.2015.03.007. Preprint (2014), doi:10.2139/ssrn.2393054.

C. Bernard, X. Jiang, and S. Vanduffel. Note on 'Improved Fréchet bounds and model-free pricing of multi-asset options' by Tankov (2011). *Journal of Applied Probability*, 49(3):866–875, 2012.

C. Bernard, Y. Liu, N. MacGillivray, and J. Zhang. Bounds on capital requirements for bivariate risk with given marginals and partial information on the dependence. *Dependence Modeling*, 1:37–53, 2013.

C. Bernard, M. Denuit, and S. Vanduffel. Measuring portfolio risk under partial dependence information. *Social Science Research Network*, 2014. doi: 10.2139/ssrn.2406377.

C. Bernard, L. Rüschendorf, and S. Vanduffel. Value-at-Risk bounds with variance constraints. *Journal of Risk and Insurance*, 2015a. doi: 10.1111/jori.12108. Preprint (2013), available at http://ssrn.com/abstract=2342068.

C. Bernard, L. Rüschendorf, S. Vanduffel, and J. Yao. How robust is the Value-at-Risk of credit risk portfolios? To appear in: *The European Journal of Finance*, 2015b. doi: 10.1080/1351847X.2015.1104370.

C. Bernard, L. Rüschendorf, S. Vanduffel, and R. Wang. Risk bounds for factor models. *Social Science Research Network*, 2016. doi: 10.2139/ssrn.2572508.

V. Bignozzi, G. Puccetti, and L. Rüschendorf. Reducing model risk via positive and negative dependence assumptions. *Insurance: Mathematics and Economics*, 61(1):17–26, 2015. doi: 10.1016/j.insmatheco.2014.11.004.

P. Deheuvels. La fonction de dépendance empirique et ses propriétés. *Académie royale de Belgique, Bulletin de la classe des sciences*, 65(5):274–292, 1979.

M. Denuit, J. Genest, and É. Marceau. Stochastic bounds on sums of dependent risks. *Insurance: Mathematics and Economics*, 25(1):85–104, 1999. doi: 10.1016/S0167-6687(99)00027-X.

M. Denuit, J. Dhaene, and C. Ribas. Does positive dependence between individual risks increase stop loss premiums. *Insurance: Mathematics and Economics*, 28: 305–308, 2001.

P. Embrechts and G. Puccetti. Bounds for functions of dependent risks. *Finance and Stochastics*, 10(3):341–352, 2006. doi: 10.1007/s00780-006-0005-5.

P. Embrechts and G. Puccetti. Bounds for the sum of dependent risks having overlapping marginals. *Journal of Multivariate Analysis*, 101(1):177–190, 2010.

P. Embrechts, A. Höing, and A. Juri. Using copulae to bound the Value-at-Risk for functions of dependent risks. *Finance and Stochastics*, 7(2):145–167, 2003.

P. Embrechts, G. Puccetti, and L. Rüschendorf. Model uncertainty and VaR aggregation. *Journal of Banking and Finance*, 37(8):2750–2764, 2013. doi: 10.1016/j.jbankfin.2013.03.014.

P. Embrechts, G. Puccetti, L. Rüschendorf, R. Wang, and A. Beleraj. An academic response to Basel 3.5. *Risk*, 2(1):25–48, 2014. doi: 10.3390/risks2010025.

P. Embrechts, B. Wang, and R. Wang. Aggregation-robustness and model uncertainty of regulatory risk measures. *Finance and Stochastics*, 19(4):763 –790, 2015. doi: 10.1007/s00780-015-0273-z.

C. Genest, K. Ghoudi, and L.-P. Rivest. A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, 82(3):543–552, 1995. doi: 10.1093/biomet/82.3.543.

C. Genest, B. Rémillard, and D. Beaudoin. Goodness-of-fit tests for copulas: A review and a power study. *Insurance: Mathematics and Economics*, 44(2):199–213, 2009. doi: 10.1016/j.insmatheco.2007.10.005.

D. Hunter. An upper bound for the probability of a union. *Journal of Applied Probability*, 13:597–603, 1976.

H. G. Kellerer. Measure theoretic versions of linear programming. *Mathematische Zeitschrift*, 198(3):367–400, 1988. doi: 10.1007/BF01184672.

M. Moscadelli. The modelling of operational risk: experience with the analysis of the data collected by the basel committee. Working Paper 517, Bank of Italy - Banking and Finance Supervision Department, 2004.

G. Puccetti and L. Rüschendorf. Bounds for joint portfolios of dependent risks. *Statistics & Risk Modeling*, 29(2):107–132, 2012a. doi: 10.1524/strm.2012.1117.

G. Puccetti and L. Rüschendorf. Computation of sharp bounds on the distribution of a function of dependent risks. *Journal of Computational and Applied Mathematics*, 236(7):1833–1840, 2012b. doi: 10.1016/j.cam.2011.10.015.

G. Puccetti and L. Rüschendorf. Sharp bounds for sums of dependent risks. *Journal of Applied Probability*, 50(1):42–53, 2013.

G. Puccetti and L. Rüschendorf. Asymptotic equivalence of conservative VaR- and ES-based capital charges. *Journal of Risk*, 16(3):3–22, 2014.

G. Puccetti, B. Wang, and R. Wang. Complete mixability and asymptotic equivalence of worst-possible VaR and ES estimates. *Insurance: Mathematics and Economics*, 53(3):821–828, 2013. doi: 10.1016/j.insmatheco.2013.09.017.

G. Puccetti, L. Rüschendorf, D. Small, and S. Vanduffel. Reduction of Value-at-Risk bounds via independence and variance information. *Forthcoming in Scandinavian Actuarial Journal*, 2015. doi: 10.1080/03461238.2015.1119717. SSRN: http://ssrn.com/abstract=2584860.

L. Rüschendorf. On one sample rank order statistics for dependent random variables. In J. Kozesnik, editor, *Transactions of the Seventh Prague Conference, Volume B*, pages 449–456. Springer, 1974. ISBN 978-90-277-0894-6.

L. Rüschendorf. Asymptotic distributions of multivariate rank order statistics. *The Annals of Statistics*, 4(5):912–923, 1976.

L. Rüschendorf. On the minimum discrimination information theorem. *Statistics & Decisions, Supplement Issue*, 1:263–283, 1984.

L. Rüschendorf. Bounds for distributions with multivariate marginals. In K. Mosler and M. Scarsini, editors, *Stochastic orders and decision under risk*, volume 19 of *IMS Lecture Notes*, pages 285–310. 1991a. ISBN 0-940600-26-9. doi: 10.1214/lnms/1215459862.

L. Rüschendorf. Fréchet bounds and their applications. In G. Dall'Aglio, S. Kotz, and G. Salinetti, editors, *Advances in Probability Distributions with Given Marginals*, volume 67 of *Mathematics and Its Applications*, pages 151–188. Springer, 1991b. ISBN 978-94-010-5534-5. doi: 10.1007/978-94-011-3466-8.

L. Rüschendorf. Comparison of multivariate risks and positive dependence. *Journal of Applied Probability*, 41:391–406, 2004.

L. Rüschendorf. Stochastic ordering of risks, influence of dependence, and a.s. constructions. In N. Balakrishnan, I. G. Bairamov, and O. L. Gebizlioglu, editors, *Advances on Models, Characterization and Applications.*, pages 19–56. Chapman and Hall/CRC, 2005. ISBN 978-0-8247-4022-1. doi: 10.1201/9781420028690.ch2.

L. Rüschendorf. *Mathematical Risk Analysis*. Springer Series in Operations Research and Financial Engineering. Springer, 2013. ISBN 978-3-642-33589-1.

W. Stute. The oscillation behavior of empirical processes: The multivariate case. *The Annals of Probability*, 12(2):361–379, 1984.

B. Wang and R. Wang. The complete mixability and convex minimization problems with monotone marginal densities. *Journal of Multivariate Analysis*, 102 (10):1344–1360, 2011. doi: 10.1016/j.jmva.2011.05.002.

R. Wang. Asymptotic bounds for the distribution of the sum of dependent random variables. *Journal of Applied Probability*, 51(3):780–798, 2014.

R. Wang, L. Peng, and J. Yang. Bounds for the sum of dependent risks and worst Value-at-Risk with monotone marginal densities. *Finance and Stochastics*, 17: 395–417, 2013. doi: 10.1016/j.jmva.2011.05.002.

R. C. Williamson and T. Downs. Probabilistic arithmetic. I. Numerical methods for calculating convolutions and dependency bounds. *International Journal of Approximate Reasoning*, 4(2):89–158, 1990. doi: 10.1016/0888-613X(90)90022-T.

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