ON THE PERCEPTION OF TIME

by

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Abstract. In this article we review scientific work and present new results on the perception of time, that is, on the feeling of time as perceived by individuals. The phenomenon of time being felt passing faster with growing age is well-known, and there are numerous interesting studies to shed light on the question why this is so. Many of these are based on studies in psychology and social sciences. Others range from symptoms of the ageing process to related symptoms of decreasing memory capacities. Again other explanations, quite different in nature from the preceding ones, involve event intensities in the life of individuals. The relative decrease of interesting new events as one grows older is seen as an important factor contributing to the feeling that time is thinned out. The last type of possible explanations can be made more explicit in a mathematical model. Quantitative conclusions about the rate of decrease of the feeling of time can be drawn, and, interestingly, without restrictive assumptions. It is shown that under this model the feeling of time is thinned out at least logarithmically. Numerical constants will depend on specific hypotheses which we discuss but the lower-bound *logarithmic* character of the thinning-out phenomenon does not depend much on these. The presented model can be generalized in several ways. In particular we prove that there are, a priori, no logical incompatibilities in a model leading to the *very same distribution of time perception* for individuals with completely different pace and style of life. Our model is built to explain long-time perception. No claim is made that the feeling of time being *thinned out* is omnipresent for very individual. However, this is typically the case and we explain why.

Keywords: Sensory information, Weber-Fechner law, time paradoxon, probabilistic modeling, logarithmic thinning, compression of time, Pascal processes.

Short running title: PERCEPTION OF TIME

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1. Scope of the paper.

The scope of this article is twofold: First, to review major concepts of approaches to the study of the perception of time, and second, to present a mathematical model. This model confirms the feeling that time passes more and more quickly. It also allows for extensions leading to insights which are, as we believe, new.

We will argue that one should distinguish between methods to study short or medium time perception compared with the perception of time over periods of several years or decades. The study of short-time perception has different motivations, ranging from scientific per-se reasons over auxiliary aspects aiming to help to understand certain phenomena in psychology and social sciences (as for example physiological functions of the brain) up to market-research oriented objectives, and many others. Here our review is far from being complete. We then pass to the domain of long(er) time perception and review what is known. Indeed, less seems to be known this field, and we will try to summarize the reasons why this has to be expected. But then, taking these reasons as a new repère, we conclude that, viewing long-time perception, we should favour models which are flexible and as general as possible. We then present our own mathematical model and results, which we see as first modest steps into this direction.

2. Perception and perception of time.

Perception is generally understood as the process of interpreting sensory information. It is considered as one of the classical notions in psychology and cognitive sciences. The Weber-Fechner law of perception, to which we will return in more detail, is seen by many scientists as arguably the most important tool to understand the notion of perception, at least for many types of perception. However, there are counterexamples. To give just one example, the meaning of this law is less evident in esthetic perception, as e.g. the perception of beauty (Bösel [1]).

The perception of time seems different in nature from what we usually understand as perception. It seems to have its own ways and own laws. Interestingly, understanding these laws seem to be subtle. Is it possible to really understand the perception of time?

Fraisse (1911–1996) was convinced that it is possible ([2]). It is no exaggeration to say that he has devoted his scientific life to this question. He also stimulated research in several new directions. Einstein, however, may have had a different point of view. According to Buccheri [3], Einstein is reported to have said that the feeling of time is beyond scientific enquiry since time is reversible such that "there is no *now.*" – We dare to add here that it would be informative to know in which context Einstein was asked, because the answer, as it stands, seems non-self-explanatory.

Prigogine [4], in contrast, stayed faithful to the idea of time being, in its essence, irreversible. He believed in the necessity to update physics by allowing for the concept of an intrinsic irreversible time giving rise to the unpredictability of the future. The latter appeal is compatible with Barbour's [5] understanding that the human illusion of the flow of time can be derived from a collection of short-time inter-related images of the reality, because, as we conclude, in this illusion the future cannot play but a limited role. Kozyrev goes even further than Barbour by proposing the existence of a new physical entity termed "time flow" which can neither be identified with space, nor field nor matter (see e.g. [6]).

For an individual, the feeling of time, can clearly not be dissociated from the concept of "cognition" or concepts of "stimuli" of perception (Pöppel et al. [7]), or concepts of choice and selectivity (Carstensen et al. [8]), and more generally, from a subjective experience of a changing world.

Our own point of view concurs with these ideas in the sense that we find it difficult to imagine an individual perceiving time without being submitted to agents which actually cause perception. (See also Vicario [9]). Stimuli and sensory information should be at the heart of a realistic study of the perception of time.

2.1 Different types of sensory information. Different types of sensory information cannot be considered on the same footing, and therefore scientic methods show naturally a great variety. Compared with empirical studies of the perception of primary types of sensory information, like for example temperature, loudness, etc., empirical studies of the perception of time are more difficult. The reason is that we cannot control physical time because we cannot stop it or increase or decrease its speed. For pressure we can work with increments (+ or -) of pressure, dp, or temperature, $d\tau$ and we can learn from the impact of those increments. The idea of arguing in terms of increments is fundamental in science. Newton, despite his genius, would probably not have discovered the laws of gravitation if nobody had been able before to make experiments in order to estimate the acceleration of a stone in free fall. To come back to Einstein we may conclude that he should be considered, in that respect, as a remarkable partial exception to the rule.

Having said this, certain empirical studies can be done for estimating the perception of time, of course, at least to some extent. For example one can submit a test person to a short-time test of a few seconds, say, and then ask: How long do you think it took? This is an indirect approach. We cannot vary the speed of time but can assess increments of perception by varying the *length* of time. There are several examples cited in the literature which refer to such short-time experiments. However, this does typically not work for long-time perception. Indeed, it is amusing to imagine somebody would ask us: Look back exactly 40 years. Estimate how long it took since then until today? Hence we must see studies of time perception, when time increases, as diverging from its short(er)-time analogue.

3. Short(er)-time perception.

Many interesting phenomena have been discovered in this domain. Fasolo et al. [10] have found that the estimation of the (physical) time used to make a decision on a set of possible choices is affected by the number of options in this set. Test persons who had many options had the tendency to underestimate the physical time they spent to take the decision whereas for test persons having few options, the contrary was the case. Glicksohn [11] discusses the influence of altered sensory environments, and Wittman and Paulus [12] the impact of a process of decision making.

Research on intertemporal decisions indicate that most people are biased towards the present (O'Donoghue and Rabin [13], Thaler[14], and Zauberman and Lynch [15]). Future events with small and moderate horizon are often individually discounted, in particular those which involve financial implications. Therefore the perception of time plays also an important role in consumer research (Graham [16]). Information about how people see and/or feel time can be gained from studying their discounting tendencies. Read et al. [17], Rubinstein [18], and Zauberman et al. [19] explain several such tendencies and stress the importance of *discounting time*, hyperbolic discounting in particular (see also Ainslie [20], and Ainslie and Haslam [21]). Ariely and Loewenstein [22] look at another interesting side of the question, namely how time matters in judgement. The question of impatience is also connected with the perception of present time as explained in Scholten and Read [23].

4. Factors governing the feeling of time.

As indicated before it is generally accepted that there seems to be no clear proportionality factor between time length as felt by an individual and the actual length of real time periods. Does the perception of time depend on situations as well as circumstances? Many researchers would agree that this is the case. But then, why is this so, and furthermore, to what extent is it true?

Many personal experiences as well as experimental studies confirm that situations, and circumstances under which they occur, play an important role. Happy hours for instance are perceived by an individual as passing fast, but twenty minutes of waiting for a bus appears long, and one minute of pain much longer. The perception of time in a given period seems closely connected with the number of new, unusual or remarkable events which take place in this period (see e.g. James [24] and Block and Zakay [25]). Periods which are filled with new things are momentarily seen as passing by quickly. Looking backwards they have made an impression, and now they seem much longer than less exciting periods of life. To give an example, many people would agree that the very first days of a vacation are well remembered whereas the days or weeks thereafter seem to have passed more discreetly or even in a virtually imperceptible way. For events the aspect of the *new* seems to make the difference for the posterior perception of time rather than the relative length of time which it took to live the new event within its period.

It has also been documented by several independent tests, that, as one would intuitively expect, "interesting" time periods pass by more quickly. So for instance, showing entertaining movie clips during ten minutes to a group of probands was felt much shorter than ten minutes filled with some sort of routine work. Consequently, time has been coined as a "dimension of perception and experience". This is somewhat vague, because it is not clear in which way these two components are supposed to collaterate. Nevertheless it seems safe to say that the perception of time is intrinsically connected with new events which are experienced and serve as orientation.

5. The time paradoxon.

There is an interesting phenomenon which is ususally called *time paradoxon* in the literature of psychology, and which is relevant for understanding the individual perception of time. We have outlined one side of this already as an observation, when we compared the impact of the first days of a vacation with the one of subsequent days. But there is more to it. While time periods which are filled with interesting activities pass by fast, these periods are felt in retrospection as having taken longer than less eventful periods. Hence, in retrospective, the feeling of time duration is in general different from its perception at the time (instant) of the very same period. One convincing explanation of this is that human beings remember, first of all, *major* events of their life. Periods of these major events are memorized in a particular way and leave an accessible track on the human mind. Moreover, the meaning of a major event changes naturally in time. A first event of a certain type has a greater chance to be felt as *major* than similar events later on in life. Therefore a month in childhood or adolescence is usually felt much longer than a month in adult age. The feeling of time is thinned out in a quite natural way. On the whole the phenomenon seems almost unavoidable. Taking these arguments together gives additional support to the idea that important or new events and their pattern of occurrence in life play a dominant role for the individual perception of time.

6. Perception and the Weber–Fechner Law.

The Weber–Fechner Law is fundamental in the general theory of perception in psychology. It states that the excitation released by sensory stimuli is proportional to the logarithm of the magnitude of the stimulus. See e.g. Dehaene [26]. This law was first discovered empirically by the physiologist Weber and then later deduced in a mathematical form by the physicist Fechner. Fechner [27] started from the assumption, that the necessary change of the stimulus' magnitude to reach a perceptible difference in the excitation is proportinal to the magnitude of the *initial* impulse. In one of his experiments, Weber [28] gradually increased the weight that a blindfolded man was holding and asked him to respond when he first felt the increase. Weber found that the smallest noticeable difference in weight – which means the least difference that the test person can still perceive as a difference – was proportional to the starting value of the weight. In mathematical terms this statement can be written as a simple differential equation, namely $dp = k\ dS/S$, where p stands for perception, dp for the differential change in perception, dS for the differential increase in the stimulus, and S for the stimulus. The factor k is a constant which can be determined experimentally. Integrating the above equation gives $p = k \log S + c$, Here c is the constant of integration. To determine c, put p = 0, i.e. there is no perception. Then $c = -k \log S_0$, where S_0 is that stimulus threshow below which there is no perception. Taking these equations together yields $p = k \log S - k \log S_0$, that is

$$p = k \log \frac{S}{S_0}. \tag{1}$$

The relationship between perception and stimulus is thus logarithmic. This means that if a stimulus is multiplied by a fixed factor then the corresponding perception is altered by an additive constant. In other words, for multiplications in stimulus magnitude, the strength of perception only adds. Relationship (1) has been seen to be valid not just for the sensation of weight but also for other stimuli of sensory perceptions. However, as far as we know, not for the perception of time.

7. New results on the preception of time.

Bruss and Rüschendorf [29] proposed a mathematical model to assess the quantitative behaviour of the individual perception of time. This model is based on the hypothesis that the perception of time is proportional to the number of new events or important events. Since the definition of *event* may differ greatly from one individual to another, the objective was to create a model which is sufficiently simple to cope with the need of general acceptance and with the desire to obtain a quantitative assessment.

The result we obtained by this model is a law for the subjective feeling of time. This law states that the peception of time is thinned out on a logarithmic scale. The logarithm is (for every reasonable basis) a concave shaped increasing function, i.e. the older we get, the more our actual feeling of time is thinned out for periods of the same length. The period of the decade from age ten to twenty seems longer that the decade from age fifty to sixty. (Exceptions from the rule are addressed in the Discussion Section.)

This logarithmic law of thinning is similar to the Weber-Fechner law for sensory perception. The discovered parallele is surprising in the sense that our assumptions have not much in common with those of Fechner. In particular, there is no notion of an initial stimulus or an initial magnitude in our model. However, one should also note that this parallele shows a certain consistency. After all, if we speak of the perception of time then we imply that we *perceive.*

8. The Bruss–Rüschendorf model.

The starting point of our model is the discretization of time and a concurrent discretization of events. Our whole life is supposed to host N different events, or more precisely, N different event types, because many events are repetitive. This N is most likely different for all of us, and unknown to each of us. It is most likely different for multiple reasons. First of all, we may have diverging notions of when to call an event "event", and when we would speak of a significant event. Then also, we have different lifestyles implying different frequencies and patterns of events in time. In particular, we do not all reach the same age. Finally, there is a good and fortunate reason that we all ignore our N. Our Nth new event is our last one – death.

It may come as a surprise that the exact definition of an event is, as we shall show, of little importance. Our conclusion is almost independent of such a definition. For this reason we can take the liberty to see events as "lump events", that is, as self-contained units, although in real life certain events may be confounded. Also, the order of magnitude of N, be it in the hundreds or many thousands, plays, as shown below, a role of minor importance.

Our model is a box model. We imagine N small boxes standing for events. We need not think of the boxes of being arranged in some order. Balls are now placed into these boxes. If a ball falls at some time into a box marked by an event we interpret this by saying that the corresponding event occurs at that time. The chronological course of our life is now seen as the order in which the balls are placed into the boxes. Those boxes which are still empty at a given time are those events which have not occurred so far. Boxes with one ball stand for events which have happened so far exactly once, and those with more than one or even many balls stand for events we have experienced more than once. Each box will be finally filled. We know that the last ball will go into the special box number N, which is by definition the last empty box.

8.1 Implications of the model. Our assumption that the new events are dominant for the feeling of time is now translated correspondingly: If in a given time interval the number of placements in empty boxes is large, then this is interpreted as a high stimulus for the perception of time. If it is smaller, the preception of time is weaker. To keep the model simple, we measure the impact on the perception by the relative frequency of *first* placements into empty boxes with respect to the total number of balls placed in this time interval.

How many balls are needed until the Mth, say, different event occurs, that is, until M boxes are filled? The answer depends on N. For instance, if M is larger than N, then there is no way. Also, the bigger N, the more likely it becomes that initial placements go into new boxes only. Let us therefore consider for a moment N as being large and fixed, and M as fixed between 1 and N. Obviously it takes at least M to fill M boxes, but often it will take longer, because there may be many repetitive events. The answer depends furthermore on the mode of allocation. With the generosity we allowed for by not specifying what we mean by an event, this number can theoretically be infinite, if $M > 1$. Some boxes will be quickly filled, like for instance eating a standard lunch, or brushing teeths. These repeated events are here supposed to leave no impact on the feeling of time. But some other boxes may have to wait a long time. For instance, we would not object the Nth box to have to wait for a long long time to receive the ball.

8.2 The mode of allocation. Let us first suppose, that every ball falls, independently of other balls, equally likely in any of the N boxes. This is a simple and seemingly restrictive assumption and we we will study its impact in more detail later on. For this model a straightforward probability calculation shows that the expected number of balls required to fill M urns with at least one ball equals

$$1 + \frac{N}{N-1} + \frac{N}{N-2} + \cdots + \frac{N}{N-M+1}, \quad 1 \le M \le N. \tag{2}$$

The sum in (2) is well approximated by

$$A(N, M) = N \log \left(\frac{N}{N-M+1} \right), \tag{3}$$

where log in (3) denotes the logarithm to the basis $e = 2.718281 \cdots$. Although we do not know N, this approximation contains already much information. For the same increment of *perception of time* it is the ratio $A(N, M)/N$ which counts, and this is $\log(N/(N-M+1)) = \log(1 + (M-1)/(N-M+1))$, which can only grow in the process of life. Physical time is the time needed to fill the boxes by balls, but it is the occurrence of new boxes which determine the perception of time, requiring on average an ever increasing number of balls. This means that the physical time would have to accelerate to create the same increment of time perception. The speed of real time is constant, however. Hence our perception of time is bound to decrease with respect to fixed increments of real time. This is what we feel.

8.3 Logarithmic thinning or harmonic acceleration? We interpret the above equation for $A(N, M)$ as a *logarithmic thinning* of time in our model, and we will use this terminology throughout this paper. We should underline that we could interpret the result the other way round. The content of the statement itself would be the same, however. It is true that the multiplying factor $\log(N/(N-M+1)$ is becoming larger as M becomes larger, so that one could say that the (physical) time accelerates, and compute the corresponding acceleration. For fixed large N and $t := M/N$ we have $A(M, N) \sim a(t, N)$, where

$$a(t, N) = N \log \left(\frac{1}{1 - t + \frac{1}{N}} \right) = -N \log(1 - t + \frac{1}{N}). \tag{4}$$

Since acceleration is the derivative of speed, (4) implies that the acceleration at time t is given by

$$\frac{\partial a(t, N)}{\partial t} = \frac{N}{(1 - t + \frac{1}{N})} \sim \frac{N}{1-t}. \tag{5}$$

Hence this is what one may call (reverse)-harmonic acceleration. However, if we speak according to (5) of acceleration then we imply that the physically measurable time accelerates, whereas we consider the latter to have constant speed. Hence, to stay consistent, we say that the perception is thinned out with respect to real time. In other words, we use more and more real time to get an equal increment of preception of time. This is really what is meant when people say "time flies, and this more and more quickly."

At the beginning the effect is hardly noticeable. For small M and large N, the expected acceleration of real time with respect to the perceived time, $\log(1+(M-1)/(N-M+1))$, is only little larger than 0, i.e. the relative feeling of time compared to its real counterpart is almost one. But as time goes on, M will exceed one tenth, one quarter, one half etc of N. The jump from $M = 0.2N$ to $M = 0.4N$, say, is the same increase of perception as the jump from $M = 0.6N$ to $M = 0.8N$, but calculation shows the real time has to go up by a factor of about 2.4. In order to fill half of the N boxes, only about $h(N) = 0.7N$ balls are needed, but to fill them all needs on average about $N \log(N)$.

8.4 Possible bias. We see two instances of possible bias in our model.

The first concerns the lack of the notion of "magnitude" of an event in our model. Events may be of minor, medium or great importance in the life of an individual. It is the important events which are of most concern. If there is for instance one outstanding event in life compared to which *all* others seem negligible, then hypothesis that the feeling of time is proportional to the number of *new* events is hardly convincing after the occurence of the outstanding event. We are fully aware of this weekness. On the other hand we believe that it makes little sense to try to model the distribution of important events in life of an individual. There are too many unknowns. In other words, we see no justification in trying to adapt the model such that individual thinning can be taken into account.

The second is the end of life. New events become very rare at the end, as the expression $N \log N$ shows. The latter may seem like too much thinning, in other words, the waiting time for seeing the last empty boxes filled becomes too large in expectation, in particular if we consider N as very large. Now, close to the end of lifetime any probabilistic box model must have some default. Most boxes become clearly inacessible shortly before death, whereas other boxes become, unfortunately, almost compulsary and will fill up in a quasi-automatic order. However, this bias becomes usually only a problem for M being very very close to M. Hence, unlike in the first case, we would not see this as a weekness of the model.

9. Weakening the hypotheses

Before drawing further conclusions from our model it is now time to scrutinize our major hypotheses. First, is the condition that balls are placed independently of each other crucial for our model? Second, what is the influence of the unrealistic hypothesis that all new events (except at the very end) have equal probability?

9.1 Independence. It is clear that this hypothesis cannot hold throughout. Suppose for instance that a ball has just been placed in a box marked "car break-down". If the box model contains another box marked "buy new car", say, then this box will in the near future receive a ball with an increased probability. Hence placements in these boxes (events) are not independent. Still, they are arguably independent of most other events which have little to do with car problems and implications. When introducing the model we pointed out that we leave open, how an event should be defined exactly, so that we may also think of a "car event" box which receives all the "related" balls. This re-definition affects the number of boxes n, but as we have seen in Subsection 8.2 this has not much impact on the central factor $N/(N-M+1)$, because M and N are submitted to the same change of scale. Thus the independence hypothesis can be reasonably well defended. For the majority of individuals certainly not as restrictive for the conclusion as it first may first seem.

There are of course exceptions. For instance, if a person is for most of his or her life so seriously ill that most events dominated by this illness then the overall-dependence becomes too stong and the model becomes irrealistic.

9.2 Equiprobable boxes. Boxes are equiprobable if the random placement of a ball is for each ball uniform over all boxes. This assumption is probably not realistic. It is convenient because computations become simple. However, the true distributions, as well as the definitions of boxes, must be expected to vary from one individual to another and it would be equally irrealistic to suppose to know these. Thus, in practice, there is no real alternative to assuming the same distribution for all, although not necessarily the uniform distribution. The latter raises the most important question: Is it possible that our the assumption that all boxes are equally likly exaggerates the effect of thinning in the perception of time?

Here mathematics gives an interesting and unrefutable answer: It is No. There is in general no exaggeration. Indeed, we shall show below that for any subset of empty boxes the expected number of balls needed to fill them is minimal *if and only if* each box is, within this set, *equally likely* for any placement of a ball. Thus under another scheme of independent non-identical probabilities for different boxes, the thinning out of the perception of time cannot become but stronger. In other words, if one accepts that the feeling of time is dominated by new events, logarithmic thinning in the perception of time is a lower bound!

9.3 Logarithmic thinning is a lower bound. We have to show that, in any box scheme and for any mode of allocation of balls into the boxes, the expected time to fill all boxes in a given set is minimal if and only if all placements are equally likely in this set.

Proof: Consider n boxes and an infinite supply of balls. Imagine the boxes being numerated from 1 to n. For each placement of a ball, we suppose that the kth box is chosen with probability p_k independently of the preceding placements, and that $p_1+p_2+\cdots+p_n = 1$. Each placement is supposed to take one (physical) time unit. There is no restriction for placements, i.e. each box can receive an arbitrarily large number of balls. When a ball is placed in a box which was empty so far, then we speak of a *new event.* At time 0 all boxes are supposed to be empty.

Let $T_0 = 0$, and let T_k be the time of the kth new event. If $p_k = 1/n$ for all $k \in \{1, 2, \cdots, n\}$, then we are in the case described in Section 8. Hence from (3) with

$N = n$ and $M = 1$,

$$E(T_n) \approx n \log(n). \tag{6}$$

This is also known from the so-called coupon collector's problem (see e.g. [30]).

Now let $S_n = \{1, 2, \cdots, n\}$ and let T_{S_n} be the total time, until all n boxes of the subset S_n of boxes are occupied. Hence $T_n = T_{S_n}$. Then it is easy to see that the expectation of the time T_{S_n} must satisfy the recurrence relation

$$E(T_{S_n}) = \sum_{j \in S_n} p_j \Big(1 + E(T_{S_n - \{j\}})\Big) = 1 + \sum_{j \in S_n} p_j E(T_{S_n - \{j\}}). \tag{7}$$

We now prove that $E(T_{S_n})$ is minimal on the set $\{p_1, p_2, \cdots, p_n\}$ with $p_1 + p_2 + \cdots + p_n = 1$ if and only if $p_1 = p_2 = \cdots = p_n = 1/n$.

Our proof is by induction on n. Let $n = 2$ and $p_1 + p_2 = 1$. We can assume $0 < p_1 < 1$ and hence $0 < p_2 < 1$, since otherwise one of the boxes cannot be filled. Since the occupation times for the boxes are geometrically distributed with expected values $1/p_1$ for the first and $1/p_2$ for the second box, we get from the recurrence equation (7)

$$E(T_{S_2}) = 1 + \frac{p_1}{p_2} + \frac{p_2}{p_1}. \tag{8}$$

Now note that $(p_1 - p_2)^2 \geq 0$ which means $p_1^2 + p_2^2 \geq 2p_1p_2$. Dividing the latter inequality by p_1p_2 yields $p_1/p_2 + p_2/p_1 \geq 2$. Therefore $E(T_{S_2}) = 1 + p_1/p_2 + p_2/p_1 \geq 3$, and this minimum is obtained by $p_1 = p_2 = 1/2$. Also, substituting $p_2 = 1 - p_1$ and differentiating with respect to p_1 shows that $p_1 = p_2 = 1/2$ are the only candidates for a minimum. Hence the solution is unique. This proves the minimality statement for $n = 2$.

Suppose now, as induction hypothesis, that this statement is also true for all $2 \leq r \leq n$. From the recurrence equation (7) for $S_{n+1} = \{1, 2, \cdots, n, n+1\}$ we get

$$E(T_{S_{n+1}}) = 1 + \sum_{j=1}^{n+1} p_j E(T_{S_{n+1} - \{j\}}). \tag{9}$$

However, each set of boxes $S_{n+1} - \{j\}$ consists of n boxes. Hence we can apply the induction hypothesis to conclude that for all independent allocations of balls

$$E(T_{S_{n+1} - \{j\}}) \geq E(\tilde{T}_{S_{n+1} - \{j\}}),$$

where $\tilde{T}_{S_{n+1} - \{j\}}$ describes the total filling time for $S_{n+1} - \{j\}$ under uniform placement probabilities. The uniform placement means now a placement probability $(1 - p_j)/n$ for each box in the subset $S_{n+1} - \{j\}$. Since the expected entrance time into the set $S_{n+1} - \{j\}$ (i.e. the time to fill a box in S_{n+1} other than box j) is $1/(1 - p_j)$ it suffices to minimize the sum $\sum_{j \in S_{n+1}} p_j/(1 - p_j)$ under the sum constraint $\{p_1 + p_2 + \cdots + p_n + p_{n+1} = 1\}$. Following the Lagrange multiplier method we set

$$F(p_1, p_2, \cdots, p_n, p_{n+1}; \lambda) = \sum_{j=1}^{n+1} \frac{p_j}{1 - p_j} + \lambda \Big(1 - \sum_{j=1}^{n+1} p_j\Big). \tag{10}$$

The partial derivatives with respect to the p_j are

$$\frac{\partial F}{\partial p_j} = \frac{1}{(1-p_j)^2} - \lambda = 0, \;\; j = 1, 2, \cdots, k+1,$$

and the constraint yields

$$\frac{\partial F}{\partial \lambda} = 1 - \sum_{j=1}^{n+1} p_j = 0.$$

Since the function $f(x)$ defined by $f(x) := 1/(1-x)^2$ is bijective on $[0,1]$, there is only one p_j satisfying $f(p_j) = \lambda$. Hence all solutions p_j must be identical and their sum must be equal to one, that is, $p_j = 1/(n+1)$ for all $j \in S_{n+1}$. Further, as the objective function takes its minimum in the interior of the set $\{p_1 + p_2 + \cdots + p_n + p_{n+1} = 1\}$, the above system of equations based on the partial derivatives is bound to display all candidates for a minimum. Since there is only one such candidate, namely $p_1 = p_2 = \cdots = p_n = p_{n+1} = 1/(n+1)$, the latter must be the unique minimum, and the proof is complete.

We summarize: If the allocation mode of independent uniform placements on a given set of boxes is replaced by another independent allocation mode then filling all boxes in this set takes on the average longer, i.e. the perception of time becomes weaker. This implies that logarithmic thinning is a lower bound of thinning. And as far as we are aware, this result is new and should be highlighted:

In any independent-allocation box model in which the perception of time is proportional to the number of new events in life, this perception is thinned out at least logarithmically.

10. Extending the model in new directions

We now return again to the question of individual perception of time.

10.1 Random box setting. Some individual control over perception of time is indeed possible. (See also Klein [31] and Rammsayer [32]). Highlights in a life often induce many other events. For instance, if someome has just become a first-time father or mother this will entail entirely new situations. The same is the case, and even more so, if someone starts a new profession, a new position abroad, etc. In our model this means that certain boxes would fill almost automatically in a well-directed order. Similarly, if we consider the number N itself as a variable, this may be seen as creating many new boxes which would not have been there before, and which are added to the original N, whatever it is.

This is our extension. N need not be regarded as an unknown, fixed number but as a number which may change. For instance we may see N as a function of the current state M of occupied boxes, that is $N := N(M)$. Individuals have now control on the pattern of new events. There is no longer this aspect of predestination which lurked through the N-fixed model. Individuals can increase N by thinking of new activities, or a new lifestyle.

10.2 Compatibility. Doubts may arise whether this extended model will still make sense for our objective. Would the model cope with such an important step into generalization? Moreover, could it possibly display an overall common feeling of time for different individuals? In other words, is it realistic to expect an *invariance* of time perception (i.e. the same distribution of interesting events in time) both in the N-fixed model and in a model where N itself evolves like a self-stimulating process?

Note that the question of thinning is here, a priori, subordinated. It is the question concerning the invariance property on which we have to focus.

11. Pascal processes and the phenomenon of invariance.

As surprising as it may sound, it turns out that invariance presents no logical incompatibility. For certain random placement schemes at least there is an unlimited number of models which display exactly this phenomenon. Our observation seems to be new here, but its essence showed up already in the mathematical context of best choice problems for a random number of opportunities (Bruss [33]). To make this invariance intuitive we explain it in a simple model:

So far time was discrete and measured in terms of the number of balls placed. Now we consider a real-time interval $[0, T]$, where T is interpreted as the random total life length of a given individual, and time points $T_1 < T_2 < T_3 < \cdots$ when the first ball, second ball and so on are placed in his/her boxes. As before, some of the time points T_k will produce new events, others will just repeat earlier events. Since we want to allow for a self-stimulating creation of new boxes and balls we can no longer express new-event-probabilities in terms of ratios of the number of empty boxes compared with all boxes. This is why we consider a random arrival process of events on $[0, T]$ (random time points interpreted as placement times), some of which are marked "+" as interesting or new events. Let us first suppose that arrival-times are equally likely everywhere. If we look backwards at time t with $0 < t < T$ and see many points in $[0, t]$ we expect many in the remaining interval $[t, T]$. Similarly, if we see few points before t we expect few points in the future. In fact, we would expect a ratio $t/(T-t)$ for the number of points left of t and the number of points on the right of t.

Let p_k be the probability that the kth arrival time is marked +, i.e. there is a new event at time T_k. Interestingly, we can construct an arrival process of points such that the distribution of "+" points on $[t, T]$ is always the same, that is, no matter how many arrivals there were in $[0, t]$! Take for example $p_k = 1/k$. This example corresponds to the simple model in which, at the kth placement, $(1/k) \times 100\%$ of the accessible boxes are new. This is different from our original model, but it facilitates to see what is going on and is also of interest on its own. If a person is less active, i.e. if the total number of event points is not so large in $[0, t]$ then we also do not expect many points after t. Thus the values of k in the denominators of the p_k stay smaller, i.e. the p_k stay larger. For a more active person, the numbers of points in each interval would go up but the respective $1/k$ would become then smaller.

The point is that these opposite effects can be perfectly balanced by choosing a suitable random arrival processes. Bruss [34] showed the existence of a class of such processes and named them *Pascal processes.* These processes were then fully characterized and identified by Bruss and Rogers ([35],[36]). Pascal processes have a powerful modeling property, namely any deterministic monotone time change of a Pascal process yields again a Pascal process. Hence there are infinitely many ways to construct the desired invariance property. The invariance property itself is independent of what the feeling of time actually is. The reason is the above mentioned modeling property for monotone time changes which, of course, allows to stretch or shrink the real time of the arrival process of events.

As far as we are aware, this result, which is independent of any quantification of perception, is also new. We summarize:

Individuals with strikingly different lifestyles and/or life intensities
may still have exactly the same perception of time.

Now, we do not advertize Pascal processes as a tractable tool for all reasonable models. Also, we do not claim that this invariance property may hold for whatever mode of producing new events in a box model, and if it does, the Mathematics may become hard. Nevertheless, the very existence of the invariance property in reasonable mathematical models for the perception of time is unexpected and, a priori, not evident at all. And this is why this result seems so interesting.

12. Discussion.

As pointed out before, the feeling of time being thinned out need not be omnipresent. Major events in life may strongly entail other events, and if these occur then with an unusually high frequency then the feeling may be quite different. Usually, in the long run this will hardly influence the overall feeling for time unless such periods are substantial in length. One important such example is the feeling of *backlogs* (as we named it) or the feeling of a *compression of time,* as a colleague expressed it more adequately. Indeed, some individuals, in particular very successful-active and self-demanding individuals may have more and more projects in their life which they want to accomplish. But the speed of real time is constant, and so the effect is a feeling of *too many things at the same time,* indeed, a feeling of a *compression of time.* Hence this is a counterexample to our general conclusion. However, we may all agree that it should be seen as one displaying respectable rather than disturbing features.

Returning to our main conclusion, we should mention again the work of the celebrated psychologist William James [24]. (See also [37], [38].) James "saw that there is some law of proportionality but did not see what law it is." Kenney [39] formulated a *logarithmic time hypothesis*, the truth of which is seemingly of increasing interest in several scientific domains (Takahashi [40], McClure et al. [41]). Kenney's development of the hypothesis was, in its logic, a Weber–Fechner-type approach applied to the perception time. He was aware of this and made it clear that his approach is not a proof. Actually, he believed that no proof would be possible, seemingly unaware of the Bruss–Rüschendorf model [29]. His search for earlier references about the connection of the logarithm and the feeling of time was (like ours) unsuccessfull. Hence the Bruss–Rüschendorf model may indeed be the first to give the connection rigorous (mathematical) support. Moreover, we have improved this by proving now a *sub-logarithmic hypothesis* with logarithmic thinning is a lower bound for the thinning of time.

Klein [31] sees time as the *matter of which life consists.* He fears that in modern times individuals are exposed to an increasing risk of becoming "slaves other clocks", i.e.

of being submitted to a "time" defined by others. We share some of his apprehensions. In words of our model, society should not take a predominant influence on the way balls fall in our boxes. (See also Weitl [42]). For another mathematical model, aiming at a different objective however, see Planat [43].

We were surprised to see how much scientific interest is devoted in the literature to the notion of time and its perception, and we had to confine our interest (with a few exceptions) to what is related or somehow related with our model. Therefore, as we pointed out in Section 1., the survey part of this paper is far from being complete. Hence, there may be interesting directions of research on the perception of time of which we are unaware.

Taking at the end of this paper the liberty to express results for long-time perception in colloquial terms, we may summarize: Time flies more and more quickly and may frighten us by its ever-increasing speed. Still, this is a feature of life rather than a feature of time because life is sequential ... and just embedded in time.

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