On the ordering of option prices

Ludger Rüschendorf, University of Freiburg

The aim of this paper is to unify and extend some of the various results on the ordering of option prices in exponential semimartingale models. This is a problem of interest not only for financial applications but similar questions also arise in the ordering properties of Markovian networks or in the problem to develop consistency results for risk measures for portfolio vectors with respect to dependence orders. We consider these questions in the framework of multivariate semi-martingale models.

The following comparison result for two homogeneous Markov processes was recently established. Let X, Y be Feller processes with values in a LCCB space E, typically $E = \mathbb{R}^k$ with transition operators $S = (S_t)_{t\geq 0}$, $T = (T_t)_{t\geq 0}$ on $C_o(E)$ and infinitesimal generators A, B with domains D_A, D_B. Assume that $\mathcal{F} \subset C_o(E)$, $\mathcal{F} \subset D_A \cap D_B$. $\mathcal{F}$ generates a partial order $\leq_{\mathcal{F}}$ on $M^1(E, \mathfrak{B})$ defining $P \leq_{\mathcal{F}} Q$ if $\int f\, dQ$ for all $f \in \mathcal{F}$.

Theorem 1 (Comparison result for Markov processes, Rü (2007)) *Assume that the processes X, Y satisfy the following two conditions:*

1. *Y is 'stochastically monotone', i.e. $f \in \mathcal{F}$ implies that $S_t f \in \mathcal{F}$*

2. *For $f \in \mathcal{F}$ holds $Af \geq Bf$ $[P^{X_o}]$*

Then $T_t f \geq S_t f$ $[P^{X_o}]$, $\quad \forall t \geq 0$.

The proof of Theorem 1 uses a similar idea as in the classical result of Liggett (1985) on the association of stochastic processes. By a coupling argument it can be seen that Theorem 1 implies as particular case the Liggett result. It can be used to establish dependence ordering results but also to establish convex ordering results of interest for risk measures.

For the case of Lévy processes the stochastic monotonicity condition is obviously satisfied for several natural orders like convex, directionally convex, supermodular or increasing stochastic order. The second condition on the comparison of the infinitesimal generators is also necessary (and thus is necessary and sufficient) in this case.

Via a stochastic analysis approach based in Itô's formula and a general version of Kolmogorov's backward PDE Theorem 1 has further been generalized to the comparison of general multivariate semimartingale models to some Markovian semimartingale models in some recent joint work with J. Bergenthum (2006–2007). This work extends previous results in El Karoui, Jeanblanc-Piqué and Shreve (1998) and

Bellamy and Jeanblanc (2000) for diffusion and stochastic volatility models and of Guchin and Mordecki (2002) for one dimensional semimartingales.

Suppose that S, S^* are two continuous time semimartingales with differential local characteristics (b, c, K), (b^*, c^*, K^*) where S^* is Markovian. We denote by $\mathcal{G}(t, s) = E^*(g(S_T^*) \mid S_t^* = s)$ the propagation operator (value process) in the Markovian model. The basic role in the stochastic analysis approach to comparison theorems is played by the linking process $\mathcal{G}(t, S_t)$. This process forms a link between the value processes in the S and in the S^* models since

$$\mathcal{G}(0, s) = E^*(\mathcal{G}(S_T^*) \mid S_0^* = s) = E^* g(S_T^*) \quad \text{if } S_0^* = s$$

and $\mathcal{G}(T, S_T) = g(S_T)$.

As consequence, in order to obtain a comparison result the essential step is to establish that the linking process is a sub (resp. a super) martingale. This property implies that

$$Eg(S_T) = E\mathcal{G}(T, S_T) \geq E\mathcal{G}(0, S_0) = E^* g(S_T^*)$$

assuming that $S_0 = s$. This sub (super) martingale property is derived under conditions similar to 1., 2. in Theorem 1 in the papers mentioned above, together with several applications as to Lévy processes, stochastic volatility models and others.

In particular we develop in these papers some new methods which allow to establish in some examples the basic 'propagation of convexity property' respectively in the general case the 'propagation of ordering' property. The proof of these results makes essential use of a reduction to discrete time by the Euler approximation scheme and then using results on optimal couplings for the discrete time Markov operators. This stochastic analysis approach has also been extended to derive comparison results for path dependent options like for lookback, Asian, American, and barrier options (see Bergenthum and Rüschendorf (2007)).

References

[1] N. Bellamy and M. A. Jeanblanc. Incompleteness of markets driven by a mixed diffusion. *Finance and Stochastics* 4 (2000), 209–222

[2] J. Bergenthum. Comparison of semimartingales and Lévy processes with applications to financial mathematics. Dissertation, University of Freiburg (2005)

[3] J. Bergenthum and L. Rüschendorf. Comparison of option prices in semimartingale models. *Finance and Stochastics* 10 (2006), 222–249

[4] J. Bergenthum and L. Rüschendorf. Comparison of semimartingales and Lévy processes. *Annals of Probability* 35(1) (2007), 228–254

[5] J. Bergenthum and L. Rüschendorf. Some convex ordering criteria for Lévy processes. *Advances in Data Analysis and Classification* 1 (2007), 143–187

[6] J. Bergenthum and L. Rüschendorf. Comparison results for path-dependent options. Preprint (2007)

[7] N. El Karoui, M. Jeanblanc-Piqué, and S. E. Shreve. Robustness of the Black and Scholes formula. *Math. Finance* 8(2) (1998) 93–126

[8] A. A. Guchin and E. Mordecki. Bounds for option prices in the semimartingale market models. *Proceedings of the Steklov Mathematical Institute* 237 (2002), 73–113

[9] T. M. Liggett. *Interacting particle systems.* Springer, (1985)

[10] L. Rüschendorf. Comparison of Markov processes. To appear: *Journal Appl. Probability* (2007)