On optimal two-stopping problems

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Abstract

In this paper we consider optimal stopping of iid sequences $X_1, \dots, X_n$ in the full information case. The choice of two stopping times is allowed and the aim is to maximize the expected value of the better of the two stops. We determine asymptotically optimal strategies and the value of this problem for distributions in the domain of a max-stable distribution by reducing the problem to two more complicated one-stopping problems and solving them by Poisson-approximation. In comparison to structural results on this stopping problem available in the literature we obtain explicit approximative values in this paper.

Keywords: two stopping problem, max stable distribution, Poisson-process
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1 Introduction

For iid sequences with df F in the domain of a max-stable distribution Kennedy and Kertz (1991) determined the asymptotics of the optimal value of the stopping problem as well as the asymptotic distribution of the optimally stopped sequence. An alternative approach to results of this type based on Poisson-approximation of the embedded point processes was given in Kühne und Rüschendorf (2000).

In this paper we consider the asymptotics for the corresponding two-stopping problem under expected-value maximization. Let γ_m^n denote the class of all stopping times T of $X_1, \dots, X_n$ such that $m \le T \le n, \gamma^n = \gamma_1^n, \gamma_m = \gamma_m^n$ and let

$$V^2 = V^2(X_1, \dots, X_n) = \sup\{EX_{T_1} \vee X_{T_2}; \quad T_i \in \gamma^n, T_1 < T_2\} \tag{1.1}$$

be the (optimal) expected value of the two-stopping problem where the choice of two stopping times is allowed and $\vee$ denotes the maximum. Obviously, $V^2 = V^2(X_1, \dots, X_n) \ge V(X_1, \dots, X_n) = V$ where

$$V = V(X_1, \dots, X_n) = \sup\{EX_T; \quad T \in \gamma^n\} \tag{1.2}$$

is the value of the one-stopping problem. The two-stopping problem and more general multistop-problems were introduced in Haggstrom (1967). In this paper some structural results for the optimal stopping times are derived corresponding roughly to our Proposition 2.1. Extensions of these results to Markov sequences are given in Nikolaev (1999). The two-stopping problem has been considered in the case of Poissonian-streams in Saario and Sakaguchi (1992). In this case a differential equation for the optimal stopping curve was derived similarly to the corresponding differential equation in the one-stopping problem due to Karlin (1962), Siegmund (1967), Sakaguchi (1976) and Saario and Sakaguchi (1992).

Kennedy and Kertz (1991) proved for iid sequences (X_i) with df F in the domain of attraction of a max-stable law for the one-stopping problem the following results:

a) If $F \in D(\Lambda), \Lambda(x) = e^{-e^{-x}}, x \in \mathbb{R}^1$ and (T_n) are optimal stopping times for $X_1, \dots, X_n$ then

$$\begin{cases} \dfrac{EX_{T_n} - b_n}{a_n} & \to 0 \\ P\left(\dfrac{X_{T_n} - b_n}{a_n} \le x\right) & \to \begin{cases} 1 - \frac{1}{2}e^{-x}, & x \ge 0 \\ \frac{1}{2}e^{x}, & x < 0 \end{cases} \end{cases} \tag{1.3}$$

where a_n, b_n are the normalizing constants, ensuring convergence of maxima $\frac{M_n - b_n}{a_n} \xrightarrow{\mathcal{D}} \Lambda$.

b) If $F \in D(\Phi_\alpha), \alpha > 1, \Phi_\alpha(x) = e^{-x^{-\alpha}}, x \ge 0$ then

$$\begin{cases} \dfrac{EX_{T_n}}{a_n} & \to \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{\alpha}} \\ P\left(\dfrac{X_{T_n}}{a_n} \le x\right) & \to \begin{cases} 1 - x^{-\alpha}\frac{1}{2-1/\alpha}, & x \ge \left(\frac{\alpha}{\alpha-1}\right)^{1/\alpha} \\ \frac{\alpha}{2\alpha-1}\left(\frac{\alpha-1}{\alpha}\right)^{\frac{\alpha-1}{\alpha}} x^{\alpha-1}, & 0 < x < \left(\frac{\alpha}{\alpha-1}\right)^{1/\alpha} \end{cases} \end{cases} \tag{1.4}$$

c) If $F \in D(\Psi_\alpha), \alpha \ge 0, \Psi_\alpha(x) = e^{-(-x)^\alpha}, x < 0$, then

$$\begin{cases} \dfrac{EX_{T_n}}{a_n} & \to -\left(\frac{\alpha}{\alpha+1}\right)^{-\frac{1}{\alpha}} \\ P\left(\dfrac{X_{T_n}}{a_n} \le x\right) & \to \begin{cases} 1 - (-x)^\alpha \frac{1}{2+1/\alpha} & -\left(\frac{\alpha}{\alpha+1}\right)^{-1/\alpha} \le x < 0 \\ \left(\frac{\alpha+1}{\alpha}\right)^{\frac{\alpha+1}{\alpha}} (-x)^{-\alpha-1} & x < -\left(\frac{\alpha}{\alpha+1}\right)^{-1/\alpha} \end{cases} \end{cases} \tag{1.5}$$

In this paper we prove that the optimal two-stopping problem can be reduced to two somewhat more complicated one-stopping problems. The methods developed in [8] can be modified to apply in these cases. We derive asymptotics for the two-stopping problems corresponding to (1.3) and also construct asymptotically optimal stopping times. We shall concentrate on the case $F \in D(\Lambda)$. The other two cases can be handled similarly but the approximations obtained can be evaluated only numerically in these cases.

2 Asymptotics for the two-stopping problem

Let (X_i) be an independent sequence and let $V(X_1,\dots,X_n), V^2(X_1,\dots,X_n)$ denote the optimal value of the one- resp. two-stopping problem. We first show that the two stopping problem can be reduced to two more complicated one-stopping problems.

Proposition 2.1 *For $1 \le i \le n, n \in \mathbb{N}$, denote*

$$\begin{aligned} u^x_{n,i} &:= V(X_i,\dots,X_{n-1},X_n \vee x) = V(X_i \vee x,\dots,X_n \vee x) \\ &= \sup\{EX_T \vee x; T \in \gamma_i\}. \end{aligned} \tag{2.1}$$

Then

$$\begin{cases} T_{1,n} := \inf\left\{i \le n-1; u^{X_i}_{n,i+1} \ge V^2\left(X_{i+1},\dots,X_n\right)\right\} \\ T_{2,n} := \inf\left\{i > T_{1,n}; X_i \ge u^{X_{T_{1,n}}}_{n,i+1}\right\} \end{cases} \tag{2.2}$$

define optimal stopping times for the two-stopping problem and

$$V^2 = V^2(X_1,\dots,X_n) = V\left(u^{X_1}_{n,2},\dots,u^{X_{n-1}}_{n,n}\right). \tag{2.3}$$

Proof: By conditioning we decompose the two-stopping problem into two one-stopping problems.

$$\begin{aligned} V^2 = V^2(X_1,\dots,X_n) &= \sup\{E(X_{S_1} \vee X_{S_2}); S_i \in \gamma^n, \quad S_1 < S_2\} \\ &= \sup\{E\sup\{E\left(X_{S_1} \vee X_{S_2} | S_1, X_{S_1}\right); \\ &\qquad S_2 \in \gamma^n, S_2 > S_1\}; S_1 \in \gamma^{n-1}\}. \end{aligned}$$

For $S_1 \in \gamma^{n-1}$ holds

$$\begin{aligned} &\sup\left\{E\left(X_{S_1} \vee X_{S_2} | S_1, X_{S_1}\right); \quad S_2 \in \gamma^n, S_2 > S_1\right\} \\ &= \sup\left\{Eu^{X_{S_1}}_{n,S_1+1}; \quad S_2 \in \gamma^n, S_2 > S_1\right\}. \end{aligned}$$

To prove this equality observe that for $i_0 \in \{1,\dots,n-1\}$ with $P(S_1 = i_0) > 0$ we have conditionally given $S_1 = i_0$ and $X_{i_0} = x$

$$\begin{aligned} \sup_{S_2 \in \gamma^n : S_1 < S_2} E\left(X_{S_1} \vee X_{S_2}\right) &= \sup_{S_2 \in \gamma_{i_0+1}} E(x \vee X_{S_2}) \\ &= V\left(X_{i_0+1},\dots,X_n \vee x\right) = u^x_{n,i_0+1}. \end{aligned} \tag{2.4}$$

An optimal stopping time for this problem is given by

$$T^{i_0}_{2,n} = \inf\left\{i > i_0; \quad X_i > u^x_{n,i+1}\right\}.$$

This implies that $T_{2,n}^{S_1}$ is optimal for (2.4) and the above stated equality holds. Therefore, we obtain

$$\begin{aligned} V^2(X_1,\dots,X_n) &= \sup\left\{Eu_{n,S_1+1}^{X_{S_1}};\ S_1\in\gamma^{n-1}\right\} \\ &= V\left(u_{n,2}^{X_1},\dots,u_{n,n}^{X_{n-1}}\right). \end{aligned} \tag{2.5}$$

$T_{1,n}$ solves the stopping problem (2.5) and $T_{2,n}=T_{2,n}^{T_{1,n}}$ is optimal for (2.4) so our result follows. □

We prove that the general approach to the asymptotics in one-stopping problems in Kühne and Rüschendorf ([8], 2000) can be applied to the representation of $V^2 = V^2(X_1,\dots,X_n)$ in Proposition 2.1 to yield an explicit approximation of the optimal two-stopping value and to construct asymptotically optimal stopping times.

Theorem 2.2 *Let (X_i) be an iid sequence with distribution function F and $P^{X_1}\in D(\Lambda)$. Let (T_n^1,T_n^2) denote optimal stopping times for the two-stopping problem, then*

a) $$\frac{EX_{T_n^1}\vee X_{T_n^2}-b_n}{a_n}\to 1-\log(e-1) \tag{2.6}$$

b) Let (w_n)satisfy $n(1-F(w_n))\to e-1$ then

$$\begin{cases} \overline{T}_n^1 := \inf\{i;\quad X_i\ge w_{n-i}\} \\ \overline{T}_n^2 := \inf\left\{i>\overline{T}_n^1;\quad X_i\ge a_n\log\left(1+e^{\frac{X_{\overline{T}_n^1}-b_n}{a_n}}-\frac{i}{n}\right)+b_n\right\} \end{cases} \tag{2.7}$$

are asymptotically optimal two-stopping times.

Proof: By Proposition 2.1 we have to analyse the stopping problem $V\left(u_{n,2}^{X_1},\dots,u_{n,n}^{X_{n-1}}\right)$. Denote $Y_{n,i}=u_{n,i}^{X_{i-1}}$, $2\le i\le n$ and for $t\in[0,1], x\in\mathbb{R}^1$ define

$$\begin{aligned} f_n(t,x) :&= \frac{u_{n,[nt]\vee 2}^{xa_n+b_n}-b_n}{a_n} \\ &= V\left(\frac{X_{[nt]\vee 2}-b_n}{a_n},\dots,\frac{X_n-b_n}{a_n}\vee x\right). \end{aligned} \tag{2.8}$$

The embedded planar point process of the normalized (X_i)-sequence on $[t,1]\times\mathbb{R}^1$ converges in distribution to a Poisson-process

$$N_n' = \sum_{i=2}^n \varepsilon_{\left(\frac{i}{n},\frac{X_i-b_n}{a_n}\right)} \xrightarrow{\mathcal{D}} N' = \sum_i \varepsilon_{(\tau_i,Y_i)} \tag{2.9}$$

where the intensity of N' is given by $d\mu_t(s,y)=e^{-y}ds\ dy$ (see [10, p. 210]). By the approximation result for optimal stopping problems in Kühne and Rüschendorf ([8]).

$$V\left(\frac{X_{[nt]\vee 2}-b_n}{a_n},\dots,\frac{X_n-b_n}{a_n}\vee x\right)\to u^x(t), \tag{2.10}$$

where $u^x(t)$ is the optimal stopping curve of N' where a point x at time one is added to N' and we define $Y_{K^\tau} = x$ for $\tau \geq 1, K^\tau$ the stopping index of τ. $(u^x(t))$ is a solution of the differential equation

$$\frac{d}{dt}u^x(t) = -e^{-u^x(t)}, \quad u^x(1) = x. \tag{2.11}$$

Equation (2.11) has the unique solution

$$u^x(t) = \log(e^x + 1 - t) = f(t,x). \tag{2.12}$$

Therefore, the stopping boundaries converge $f_n(t,x) \to f(t,x)$ and by the continuous mapping theorem we conclude

$$\begin{aligned} N_n = \sum_{i=2}^n \varepsilon_{\left(\frac{i}{n}, \frac{Y_{n,i}-b_n}{a_n}\right)} &= \sum_{i=2}^n \varepsilon_{\left(\frac{i}{n}, \frac{u^{X_i}_{n,i+1}-b_n}{a_n}\right)} \\ &= \sum_{i=1}^{n-1} \varepsilon_{\left(\frac{i}{n}, f_n\left(\frac{i}{n}, \frac{X_i-b_n}{a_n}\right)\right)} \\ &\xrightarrow{\mathcal{D}} N^* := \sum_{i=1}^{\infty} \varepsilon_{(\tau_i, f(\tau_i, Y_i))}. \end{aligned} \tag{2.13}$$

By a simple calculation N^* is a Poisson process with intensity given by

$$\frac{d\mu^*(\cdot \times [x,\infty])}{d\lambda_{[0,1]}}(t) = \frac{1}{e^x + t - 1}. \tag{2.14}$$

In particular μ^* is continuous w.r.t. Lebesgue measure on $[0,1] \times \mathbb{R}^1$.

Using the point process convergence in (2.13) we next show that the stopping problem of the $(Y_{n,i})$ can be approximated by the stopping problem of the Poisson process N^*. To that purpose we have to investigate the further conditions for the approximation theorem in [8] for the optimal stopping of independent sequences. Since $Y_{n,i} \geq X_i$ the lower curve condition of [8] stating that for the optimal stopping curve u_n holds $\liminf u_n(1-\varepsilon) > -\infty, \forall \varepsilon > 0$ is fulfilled since it holds true already for the problem of stopping the sequence (X_i).

To check the uniform integrability condition for $\left\{\frac{(M_n-b_n)_+}{a_n}; n \in \mathbb{N}\right\}$ note that $f_n(0,x) - x$ is monotonically nonincreasing in x; for $x' > x$ holds

$$\begin{aligned} f_n(0,x') &= V\left(\tfrac{X_2-b_n}{a_n}, \ldots, \tfrac{X_n-b_n}{a_n} \vee x'\right) \\ &\leq V\left(\tfrac{X_2-b_n}{a_n} + x' - x, \ldots, \left(\tfrac{X_n-b_n}{a_n} + x' - x\right) \vee x\right) \\ &= V\left(\tfrac{X_2-b_n}{a_n}, \ldots, \tfrac{X_n-b_n}{a_n} \vee x\right) + x' - x \\ &= f_n(0,x) + x' - x. \end{aligned}$$

Since $f_n(0,0) \to \log 2 < 1$ there exists $n_0 \in \mathbb{N}$ such that $f_n(t,x) \leq 1 + x$ for $n \geq n_0, x \geq 0$. Therefore, for $n \geq n_0$

$$\frac{(M_n - b_n)_+}{a_n} \leq \frac{(X_1 \vee \ldots \vee X_n - b_n)_+}{a_n} + 1$$

and the uniform integrability of $\frac{(X_1 \vee \ldots \vee X_n - b_n)_+}{a_n}$ (see [8]) implies that of the normalized (Y_{ni}) sequence.

The optimal stopping curve of the limiting process N^* satisfies the differential equation

$$\begin{aligned} u'(t) &= -\int_{u(t)}^{\infty} \frac{1}{e^x + t - 1} dx \\ &= -\frac{x - \log(e^x + t - 1)}{t-1}\Big|_{u(t)}^{\infty} \\ &= \frac{u(t) - \log(e^{u(t)} + t - 1)}{t-1} \\ &= \frac{1}{1-t} \log\left(1 - \frac{1-t}{e^{u(t)}}\right). \end{aligned} \tag{2.15}$$

With $u(t) := v(t) + \log(1-t)$ one obtains

$$v'(t) = \frac{1}{1-t}\left(\log(1 - e^{-v(t)}) + 1\right). \tag{2.16}$$

A solution of (2.16) is given by $v(t) \equiv \log \frac{e}{e-1}$.

We next prove uniqueness of this solution. By the approximation theorem in [8] any limit $\widetilde{u}$ of a subsequence of (u_n) solves equation (2.15). Also for $T_n^{\geq [nt]}$ the optimal stopping time for $X_{[nt]}, \ldots, X_n$ holds (see [8])

$$\begin{aligned} &\frac{EX_{T_n \geq [nt]} - b_n}{a_n} \longrightarrow \log(1-t), \\ &\text{and} \\ &\frac{E\left(X_{[nt]} \vee \ldots \vee X_n\right) - b_n}{a_n} \longrightarrow \gamma + \log(1-t). \end{aligned} \tag{2.17}$$

Therefore, $\log(1-t) \leq \widetilde{u}(t) \leq \gamma + \log(1-t), t < 1$ i.e. $0 \leq v \leq \gamma$. Since for any $x_0 \in [0,1)$ the solution of (2.16) is uniquely determined by $y_0 = v(x_0)$ we have either of the two cases: $v \equiv \log \frac{e}{e-1}$ or $y_0 \neq \log \frac{e}{e-1}$. In the second case (2.16) is a differential equation in separate variables and with

$$\begin{cases} F(x) = \int_{x_0}^{x} \frac{1}{1-t} dt = \log(1-x_0) - \log(1-x) \\ G(y) = \int_{y_0}^{y} \frac{1}{1 + \log(1 - e^{-s})} ds \end{cases} \tag{2.18}$$

any solution v of (2.16) satisfies

$$G(v(t)) = F(t). \tag{2.19}$$

If $y_0 > \log \frac{e}{e-1}$ then from (2.16) $v' > 0$ on $[0,1]$ i.e. v is monotonically increasing. Since $\frac{1}{1+\log(1-e^{-s})}$ is bounded for $s \geq s_0$ and $\int\limits_{y_0}^{v(t)} \frac{1}{1+\log(1-e^{-s})} ds = F(t) \longrightarrow \infty$ for $t \to 1$ it follows that $v(t) \underset{t\to 1}{\longrightarrow} \infty$, a contradiction to boundedness of v. If $y_0 < \log \frac{e}{e-1}$ then analogously $v(t) \underset{t\to 1}{\longrightarrow} -\infty$. Together, we obtain uniqueness of the solution of (2.16). This implies that

$$u(t) = \log(1-t) + \log \tfrac{e}{e-1} \tag{2.20}$$

is the optimal stopping curve of N^*.

The approximation theorem in [8] therefore is applicable and implies convergence of the stopping problem based on point process convergence in (2.13). So from (2.3) we conclude

$$\frac{V^2(X_1,\dots,X_n) - b_n}{a_n} \longrightarrow u(0) = 1 - \log(e-1). \tag{2.21}$$

To prove the asymptotic optimality of the stopping times $\overline{T}_n^i$ note that

$$\begin{aligned} u^x(t) \ &\geq \ u(t) \\ &\Longleftrightarrow \log(1+e^x-t) \geq 1 - \log(e-1) + \log(1-t) \\ &\Longleftrightarrow x \geq \log(1-t) - \log(e-1). \end{aligned}$$

From the definition of w_n and of a_n, b_n one further obtains

$$\lim \frac{w_{[nt]} - b_n}{a_n} = \log(1-t) - \log(e-1).$$

This implies convergence of the thresholds of the stopping times $\overline{T}_n^i$ to those of the optimal stopping times in the limiting problem (see [8]). By the uniform integrability condition the convergence of expectations follows and the asymptotic optimality of $\overline{T}_n^i$ is proved. □

Remarks:

a) In comparison to the one-stopping problem where $\frac{EX_{T_n} - b_n}{a_n} \longrightarrow 0$ (cp. equation (1.3)) there is a considerable improvement by the two-stopping problem where $\frac{EX_{T_n^1} \vee X_{T_n^2} - b_n}{a_n} \longrightarrow 0.7649\dots$.

b) The cases where the distribution of the X_i is in the domain of max-stable laws $\Phi_\alpha, \alpha > 1$ resp. $\Psi_\alpha, \alpha \geq 0$ can be dealt with similarly.

One obtains

$$u^x(t) = x + \left(\alpha \frac{1-t}{\alpha - 1}\right)^{1/2} \quad \text{if } F \in D(\Phi_\alpha), \alpha > 1$$

$$u^x(t) = x - \left(\frac{\alpha}{1+\alpha}\right)^{-1/2} (1-t)^{-1/2} \text{ if } F \in D(\Psi_\alpha), \alpha \geq 0.$$

But the resulting differential equations for the limiting processes corresponding to (2.15) can be solved only numerically.

c) In comparison to the Poissonian two-stopping problem considered in Saario and Sakaguchi (1992) the Poisson processes used in this paper for approximation of the discrete problem have no bounded intensities but the point processes have accumulation points at the lower boundary.

□

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