# DUALITY THEOREMS FOR ASSIGNMENTS WITH UPPER BOUNDS

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## 1. Introduction

Let $(X_i, \mathcal{A}_i, P_i), i = 1, 2$ be two probability spaces. A probability $\mu$ on $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ is said to have marginals $P_1$ and $P_2$ if

$$P(A_1 \times X_2) = P_1(A_1) \qquad \text{for all } A_1 \in \mathcal{A}_1$$

and

$$P(X_1 \times A_2) = P_2(A_2) \qquad \text{for all } A_2 \in \mathcal{A}_2.$$

Let $\mathcal{M} = \{\mu \text{ on } \mathcal{A}_1 \otimes \mathcal{A}_2 : \mu \text{ is a probability with marginals } P_1 \text{ and } P_2\}$.

The measure theoretic version of the transportation problem dating back to Monge(1781) concerns $\sup_{\mu \in \mathcal{M}} \int h d\mu$ for $\mathcal{A}_1 \otimes \mathcal{A}_2$-measurable functions $h$ on $X_1 \times X_2$. A succinct history of the problem with its origin in the works of Kantorovich- Rubinštein(1958) and Wasserstěin(1969) can be found in Kellerer(1984). A general version of the duality theorem in this context is given in Ramachandran and Rüschendorf(1995). A close relative of the transportation problem, the assignment problem, leads to the nonatomic assignment model and its formulation as a linear programming problem (see Shapley and Shubik(1972)). Gretsky et. al. (1992) generalized the Shapley and Shubik "housing market" version of the assignment model to its continuous version in which a continuum of sellers each having a distinct house exchange them with a continuum of buyers. They prove a Portmanteau Theorem in the set up of compact metric spaces whose measure theoretic general version can be found in Kellerer(1984) which has been further generalized in Ramachandran and Rüschendorf(1995).

In this paper[1], we formulate a version of the nonatomic assignment model with upper bound constraints on the assignments in the form of a dominating measure. By a modification of the finitely additive approach to duality theorems as developed in Rüschendorf(1981), we obtain a general duality theorem for this model. A particular case of interest arises from economics when the dominating measure is supported on a given set $C \subset X_1 \times X_2$ (i.e., a subset of the economic agents control all activities in the market). We first establish in this setting a general duality theorem involving finitely additive measures. Using a specific variant of the problem, we then obtain the duality for a large class of functions in the context of $\sigma$-additive measures which enables us to derive explicit formulas in certain cases.

## 2. Notation and Preliminaries

We use standard measure theoretic terminology and notation (as, for instance, in Neveu(1965)). Let $(X_i, \mathcal{A}_i, P_i), i = 1, 2$ and $\mathcal{M}$ be as in the introduction. Let $\lambda$ be an arbitrary (not necessarily $\sigma$-finite) measure on the product space $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$. Consider the class

$$\mathcal{M}_\lambda = \{\mu \in \mathcal{M} : \mu \leq \lambda\}.$$

For a given measurable function $h$ on$(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ we define

$$S_\lambda(h) = \sup\{\int h d\mu : \mu \in \mathcal{M}_\lambda\}.$$

This leads to the dual definitions:

$$\begin{aligned} I(g) &= \inf\{\sum_{i=1}^{2} \int f_i dP_i : g \leq f_1 \oplus f_2, f_i \in \mathcal{L}^1(P_i)\} \\ I_\lambda(h) &= \inf\{I(g) + \int h_0 d\lambda : h_0 \geq 0, h_0 + g \geq h\} \\ &= \inf\{\sum_{i=1}^{2} \int f_i dP_i + \int h_0 d\lambda : h_0 \geq 0, f_i \in \mathcal{L}^1(P_i), h \leq h_0 + \sum_{i=1}^{2} f_i\}. \end{aligned}$$

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We seek to establish the duality

$$(D) \qquad S_\lambda(h) \quad = \quad I_\lambda(h)$$

for a large class of functions.

## 3. Main Results

Let $\mathcal{F} = \{\sum_{i=1}^{2} f_i : f_i \in \mathcal{L}^1(P_i)\} = \oplus_i \mathcal{L}^1(P_i)$ and let

$$\mathcal{L}_m \;=\; \{\varphi \in \mathcal{L}(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2) \;:\; \exists f \in \mathcal{F} \text{ with } f \leq \varphi\}.$$

Note that $\mathcal{L}_m$ contains all bounded measurable functions and that $\mathcal{F}$ is a linear subspace of $\mathcal{L}_m$. Let $T : \mathcal{F} \to \mathcal{R}$ be the linear functional defined by $T(\oplus_i f_i) = \sum_{i=1}^{2} \int f_i dP_i$. Let

$$\tilde{\mathcal{M}}_\lambda = \{\tilde{\mu} \;\in\; ba(P_1, P_2) : \tilde{\mu} \leq \lambda\}$$

where $ba(P_1, P_2)$ is the collection of all finitely additive measures (hereafter referred to as *charges*) on the $\sigma$-algebra $\sigma(\mathcal{R})$ generated by the class $\mathcal{R}$ of measurable rectangles in $\mathcal{A}_1 \otimes \mathcal{A}_2$ whose marginals are $P_1$ and $P_2$. Then we have

**Proposition 1** *Let $\tilde{\mathcal{M}}_\lambda \neq \emptyset$. Then (a) For all $\varphi \in \mathcal{L}_m$*

$$\tilde{S}_\lambda(\varphi) = \sup\{\int \varphi d\tilde{\mu} \;:\; \tilde{\mu} \in \tilde{\mathcal{M}}_\lambda\} \;=\; I_\lambda(\varphi)$$

*(b) If $I_\lambda(\varphi) > -\infty$, then $\exists$ a $P \in \tilde{\mathcal{M}}_\lambda$ such that $\tilde{S}_\lambda(\varphi) = \int \varphi dP$.*

Proof: The assumption that $\tilde{\mathcal{M}}_\lambda \neq \emptyset$ implies that $I_\lambda$ is positive. Since $I_\lambda \mid \mathcal{F} = T$ and $I_\lambda$ is sublinear on $\mathcal{L}_m$, by the Hahn-Banach theorem, there is an extension $S$ of $T$ to $\mathcal{L}_m$ as a linear functional such that $S \leq I_\lambda$. For any linear functional $V$ on $\mathcal{L}_m$

$$V \leq I_\lambda \Leftrightarrow V \mid \mathcal{F} = T \quad \text{and} \quad V(\varphi) \leq \int \varphi d\lambda \quad \text{for all} \quad \varphi \in \mathcal{L}_m^+$$

and so $S$ has these properties as well. By the Riesz representation theorem, there exists $\tilde{\mu} \in ba(X_1 \times X_2, \sigma(\mathcal{R}))$ representing $S$. It can now be checked that $\tilde{\mu} \circ \pi_i = P_i$ for $i = 1, 2$ and that $\tilde{\mu}(\varphi) \leq \int \varphi d\lambda$ for all $\varphi \in \mathcal{L}_m^+$. It follows that $\tilde{\mu} \in \tilde{\mathcal{M}}_\lambda$.

If $\varphi \in \mathcal{L}_m$ is such that $I_\lambda(\varphi) > -\infty$ then, as a consequence of the Hahn-Banach theorem, one obtains an extension of $S$ with $\tilde{S}_\lambda(\varphi) = I_\lambda(\varphi)$. The corresponding charge in $\tilde{\mathcal{M}}_\lambda$ then yields both (a) and (b). If $I_\lambda(\varphi) = -\infty$, then $\tilde{S}_\lambda(\varphi) = -\infty$ as well and so (a) is valid generally.

Proposition 1 establishes a duality theorem in the context of charges $\tilde{\mu} \in \tilde{\mathcal{M}}_\lambda$. We now derive general duality theorems under different settings assuming $\tilde{\mathcal{M}}_\lambda \neq \emptyset$ .

**Definition 1** *$\lambda$ is said to be $\sigma$-finite on rectangles ("rechtecksnormal" according to Kellerer(1964)) if*

$$X_1 \times X_2 = \cup_{n=1}^{\infty} R_n, \quad R_n \in \mathcal{R}, \ \ \lambda(R_n) < \infty, \quad \forall n \geq 1.$$

Note that $\lambda$ is $\sigma$-finite on rectangles if its marginals are $\sigma$-finite. Further, if $\lambda$ is $\sigma$-finite on rectangles, then it follows that (see Kellerer(1964))

$$\tilde{\mathcal{M}}_\lambda \neq \emptyset \ \Leftrightarrow \ \lambda(A_1 \times A_2) \geq P_1(A_1) + P_2(A_2) - 1 \quad \text{for all} \quad A_i \in \mathcal{A}_i$$

and that $\tilde{\mathcal{M}}_\lambda = \mathcal{M}_\lambda$. Hence we have the following general duality theorem, for assignments bounded above, without any topological assumption on the marginal spaces.

**Theorem 1** *Let $(X_i, \mathcal{A}_i, P_i), i = 1, 2$ be two probability spaces and let $\lambda$ be $\sigma$-finite on rectangles. Then*

$$(D) \qquad S_\lambda(h) \quad = \quad I_\lambda(h)$$

*holds for all bounded $\mathcal{A}_1 \otimes \mathcal{A}_2$-measurable functions $h$.*

Now we let $\lambda$ be an arbitrary measure and prove duality theorems for certain subclasses of bounded, measurable functions. For the definition and properties of perfect measures see Ramachandran(1979).

**Theorem 2** *A1) If one of the spaces is perfect then*

$$S_\lambda(h) = I_\lambda(h) \quad \forall h \in \mathcal{L}^1(\mathcal{R})$$

*where $\mathcal{L}^1(\mathcal{R})$ = the set of $\tilde{\mathcal{M}}_\lambda$-integrable functions considered as charges on $\mathcal{R}$ (see [1]).*

*A2) $(X_i, \mathcal{A}_i), i = 1, 2$ are Hausdorff topological spaces and $P_i$ are Radon measures, then*

$$(i) \quad S_\lambda(h) \quad = \quad I_\lambda(h) \quad \text{for all bounded continuous functions } h$$

*and*

*(ii) If $h$ is bounded and $\lambda$ is finite then $\exists f_i^* \in \mathcal{L}^1(P_i)$ and $h^* \geq 0$ such that*

$$I_\lambda(h) = \int f_1^* dP_1 + \int f_2^* dP_2 + \int h^* d\lambda$$

*(see Proposition 2 in Gaffke and Rüschendorf(1981), Proposition 1, Theorem 5 in Rüschendorf(1981)).*

In order to extend the conclusions of Theorem 2 to more general classes of functions one needs to extend the continuity properties of $S_\lambda$ and $I_\lambda$ along the lines of Kellerer's work (see 1984) for the case without upper bounds. This appears to be considerably difficult in general. However, in the finitely additive setting we take a new approach to treat the following case.

Let $(X_i, \mathcal{A}_i, P_i), i = 1, 2$ be two probability spaces and let $C \in \mathcal{A}_1 \otimes \mathcal{A}_2$ be such that

$$\mathcal{M}_C = \{\mu \in \mathcal{M} : \mu(C) = 1\} \neq \emptyset.$$

Notice that $\mathcal{M}_C = \mathcal{M}_\lambda$ where

$$\lambda = \begin{cases} 0 & \text{on } C^C \cap (\mathcal{A}_1 \otimes \mathcal{A}_2) \\ \infty & \text{on } C \cap (\mathcal{A}_1 \otimes \mathcal{A}_2) \end{cases}$$

With $\lambda$ as defined above, we denote by $S_C(h)$ and $I_C(h)$ the corresponding $S_\lambda(h)$ and $I_\lambda(h)$; thus

$$\begin{aligned} S_C(h) &= \sup\{\int h d\mu : \mu \in \mathcal{M}_C\} \\ &\leq \sup\{\int h 1_C d\mu : \mu \in \mathcal{M}\} \\ &= S(h 1_C) \end{aligned}$$

$$\begin{aligned} I_C(h) &= \inf\{\sum_1^2 \int f_i dP_i : \sum f_i(x_i) \geq h(x_1, x_2) \ \ \forall (x_1, x_2) \in C\} \\ &\leq \inf\{\sum_1^2 \int f_i dP_i : \sum f_i(x_i) \geq h 1_C(x_1, x_2)\} \\ &= I(h 1_C). \end{aligned}$$

We know that $S_C(h) \leq I_C(h)$ and $S(h1_C) = I(h1_C)$ (see Kellerer(1984), Ramachandran and Rüschendorf(1995)) for all bounded, measurable functions. From Theorem 2 we have

$$(D_C) \qquad\qquad S_C(h) \quad = \quad I_C(h)$$

for the classes of functions defined therein.

We now introduce a modified assignment problem. Define

$$\tilde{\mathcal{M}}_C^{\leq} = \{\tilde{\mu} \in ba(Q_1, Q_2) \quad : \quad Q_i \leq P_i, \ \ \tilde{\mu}(C^c) = 0\}$$

and let

$$\tilde{S}_C^{\leq}(h) = \sup\{\int h d\tilde{\mu} : \tilde{\mu} \in \tilde{\mathcal{M}}_C^{\leq} \ \ \} \ .$$

In economic applications where $X_1$ represents sellers and $X_2$ represents buyers this allows all feasible assignments for any subpopulations of the sellers and the buyers concentrated on the subset $C$ of $X_1 \times X_2$. Also define

$$I_C^{\leq}(h) = I(h^+ 1_C) \quad \textit{where} \quad h^+ \;=\; (h \vee 0).$$

Then we obviously have $I_C^{\leq}(h) = I_C^{\leq}(h^+)$.

That the modified problem is, in general, different from the original problem can be seen from the following example:

**Example 1** *Let* $X_i = [0,1], \mathcal{A}_i =$ *the Borel* $\sigma-$*algebra and* $P_i = \lambda$, *the Lebesgue measure, for* $i = 1, 2$. *Consider* $h^* = -2 1_\Delta$ *where* $\Delta = \{(x,x) : x \in [0,1]\}$ = *the diagonal. Then with* $C = \Delta$ *and taking* $f_1 = f_2 = -1$ *we have*

$$S_C(h) = -2 = I_C(h^*).$$

*However* $I(h^* 1_C) = 0$; *since* $g_1(x_1) + g_2(x_2) \geq h^* 1_C(x_1, x_2) = -2 1_\Delta$ *implies that* $\int g_1 dP_1 + \int g_2 dP_2 = \int (g_1 + g_2) d\lambda^2 \geq 0$ *where* $\lambda^2$ *is the Lebesgue measure in* $[0,1] \times [0,1]$.

We now proceed to establish equality and the duality for nonnegative functions for this modified problem. We need

**Lemma 1** $I_C^{\leq}$ *is a subadditive functional on* $\mathcal{L}_m$ *with*

$(a) I_C^{\leq} \geq 0$

$(b) f \in \mathcal{F}, f \geq 0 \Rightarrow I_C^{\leq}(f) = \sum_{i=1}^2 \int f_i dP_i$

$(c) I_C^{\leq}(1) = 1$, *and*

$(d) h 1_C = 0 \Rightarrow I_C^{\leq}(h) = 0$.

Proof: $(a)$ For $h \geq 0, I_C^{\leq}(h) = I(h 1_C) \geq I(0) = 0$.

$(b)$ If $\sum_i g_i \geq f 1_C$ then $\sum g_i \geq \sum f_i$ on $C$ and so $\sum \int g_i dP_i \geq \sum \int f_i dP_i$. This implies that $I_C^{\leq}(f) = \sum \int f_i dP_i$. $(c)$ and $(d)$ are obvious.

**Lemma 2** *Let* $S$ *be a linear functional on* $\mathcal{L}_m$. *Then*

$$S \leq I_C^{\leq} \Leftrightarrow \begin{array}{lll} (a) & S \geq 0 & \\ (b) & f \geq 0, f \in \mathcal{F} \Rightarrow & S(f) \leq \sum_i \int f_i dP_i \\ (c) & h 1_C = 0 \Rightarrow & S(h) = 0. \end{array}$$

Proof: "$\Rightarrow$" $(a) h \geq 0 \Rightarrow -h \leq 0 \Rightarrow -S(h) = S(-h) \leq I(0) = 0$, i.e., $S(h) \geq 0$.

$(b) f \in \mathcal{F}, f \geq 0 \Rightarrow S(f) \leq I_C^{\leq}(f) = \sum_i \int f_i dP_i$.

$(c) h 1_C = 0 \Rightarrow S(h) \leq I(h^+ 1_C) = I(0) = 0$ and $-S(h) = S(-h) \leq I((-h)^+ 1_C) = I(0) = 0$.

"$\Leftarrow$" Let $\sum_i g_i \geq h^+ 1_C \geq 0$. Then $S(h) = S(h 1_C) \leq S(h^+ 1_C) \leq S(\sum_i g_i) \leq \sum_i \int g_i dP_i$. Hence $S(h) \leq I_C^{\leq}(h)$.

As a consequence we now obtain the interesting duality theorem:

**Theorem 3** *For all $h \in \mathcal{L}_m$ we have the duality*

$$\tilde{S}_C^{\leqq}(h) \quad = \quad I_C^{\leqq}(h)$$

*and the supremum is attained.*

Proof: This follows from the Hahn-Banach theorem using the preceding two lemmas and the Riesz representation theorem as in Proposition 1.

If $h$ is bounded then the infimum is attained as well (see Rüschendorf(1981)). As a consequence we get

**Corollary 1** *For $h \in \mathcal{L}_m$, we have*

$$\tilde{S}_C^{\leqq}(h) = \tilde{S}_C^{\leqq}(h^+) = I(h^+ 1_C) = I_C^{\leqq}(h).$$

Proof: Obviously, $\tilde{S}_C^{\leqq}(h) \leq \tilde{S}_C^{\leqq}(h^+)$. Conversely, for any $\tilde{\mu} \in \tilde{\mathcal{M}}_C^{\leqq}$ letting $A = \{h \geq 0\}$ and $\tilde{\mu}_A = \tilde{\mu}|A$ we get $\tilde{\mu}_A \in \tilde{\mathcal{M}}_C^{\leqq}$ and $\int h^+ d\tilde{\mu} = \int_A h^+ d\tilde{\mu} = \int h d\tilde{\mu}_A$. This implies $\tilde{S}_C^{\leqq}(h) = \tilde{S}_C^{\leqq}(h^+)$.

We now seek to replace $\tilde{\mathcal{M}}_C^{\leqq}(P_1, P_2)$ by $\mathcal{M}_C^{\leqq}(P_1, P_2)$ consisting of the $\sigma-$additive measures. Let $(X_i, \mathcal{A}_i), i = 1, 2$ be two Hausdorff topological spaces with Radon probabilities $P_i, i = 1, 2$ and let $C \subset X_1 \times X_2$ be a closed set. Let $h \in \mathcal{L}_m, h \geq 0$; then, as in Theorem 2,

$$S_C^{\leqq}(h) = I_C^{\leqq}(h)$$

for bounded, continuous $h$ or $h$ as a uniform limit of functions of the form $\sum \alpha_j 1_{A_j \times B_j}$.

Let $\mathcal{G}^+$ ($\mathcal{F}^+$) denote the nonnegative, lower (upper) semicontinuous functions in $\mathcal{L}_m$, and let $\mathcal{R}^+$ denote the nonnegative elements in $\mathcal{L}_m$ which are increasing limits of functions of the form $\sum \alpha_j 1_{A_j \times B_j}$. Then we have

**Theorem 4** *For $h \in \mathcal{G}^+ \cup \mathcal{R}^+ \cup \mathcal{F}^+$*

$$S_C^{\leqq}(h) = I_C^{\leqq}(h) = I(h 1_C).$$

Proof: Consider $0 \leq h_n \uparrow h$, $h_n$ bounded, continuous or in $\mathcal{R}^+$, where $h \in \mathcal{G}^+ \cup \mathcal{R}^+$. Then $S_C^{\leqq}(h_n) \uparrow S_C^{\leqq}(h)$ and so we obtain from the continuity of $I$ (see Kellerer(1984))

$$S_C^{\leqq}(h) = \lim S_C^{\leqq}(h_n) = \lim I(h_n 1_C) = I(h 1_C).$$

For $h \in \mathcal{F}^{+b}$ (elements of $\mathcal{F}^+$ bounded above), let $h_n$ be bounded, continuous functions with $h_n \downarrow h$. Then, by similar arguments as in Proposition 1.26 of Kellerer(1984), $S_C^{\leqq}$ is continuous downwards on $\mathcal{F}^b$. The continuity of $I$ and the argument in Proposition 2.3 of Kellerer(1984) imply the result.

Extensions of this duality theorem to the class of nonnegative measurable functions in $\mathcal{L}_m$ as well as under constrained marginals will be given in a subsequent paper. Note that, in the present setup, for $h = 1_B$ the dual functional has a wellknown explicit representation

$$I_C^{\leq}(1_B) = I(1_{B\cap C}) = \inf\{P_1(A_1)+P_2(A_2) \ : \ B\cap C \subset (A_1\times X_2)\cup(X_1\times A_2)\} \ .$$

Thus we have the above explicit formula for closed or open sets $B$ for the assignment problem concentrated on a subset $C$ of $X_1 \times X_2$.

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