Ein Martingalansatz zum optimalen Stoppen

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Contents

Some Fundamentals

<u>Definition of SBM</u>

A continuous stochastic process $X_t,\ t \geq 0$ is a standard Brownian motion with start in x, if

i) $X_0 = x$

ii) For $t_0 < t_1 < \ldots < t_k$ the random variables $X_{t_{i+1}} - X_{t_i},\ i = 0, \ldots, k-1$ are independent.

iii) For $s < t$ $X_t - X_s$ is distributed according to $N(0, t-s)$.

Note: $E(X_t) = EX_0 = x$ and $\mathrm{Var}(X_t) = t$.

<u>Definition of Brownian motion</u>

A continuous stochastic process $X_t,\ t \geq 0$ is a Brownian motion with starting point x drift μ and variance σ^2 if there exists a standard Brownian motion $B_t,\ t \geq 0$ such that

i) $X_0 = x$

ii) $X_t = \sigma \cdot B_t + \mu \cdot t$

Definition of a Martingale:

A stochastic process $X_t,\ t \geq 0$ is a martingale with respect to the filtration $\mathcal{F}_t,\ t \geq 0$ if

i) X_t is $\mathcal{F}_t$-measurable

ii) For $s < t$ $E(X_t \mid \mathcal{F}_s) = X_s$

A consequence:

$$E(X_t) = E(X_0)$$

Theorem (Lévy):

A continuous martingale $X_t,\ t \geq 0$ with $E(X_t^2) < \infty$ for $t \geq 0$ is a standard Brownian motion if $E\left((X_t - X_s)^2 \,|\, \mathcal{F}_t\right) = t - s$ holds for all $s < t$.

Definition:

A continuous stochastic process $(X_t, \mathcal{F}_t; t \geq 0)$ is called a diffusion with start in x_0 if there exists a standard Brownian motion $(B_t, \mathcal{F}_t; t \geq 0)$ such that

$$(+) \qquad X_t = \int_0^t a(X_s)ds + \int_0^t \sigma(X_s)dB_s$$

with $X_0 = x_0$ and $a : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}_+$ measurable.

Remark:.

(+) is usually expressed as SDE:

$$dX_t = a(X_s)dt + \sigma(X_s)dB_s.$$

Examples of Optimal Stopping

a) Detection of a Trend Change

$B_t,\ t \geq 0$ standard Brownian motion

τ a random time

Let $W_t = \begin{cases} B_t & \text{for} \quad t < \tau \\ B_t + \theta(t-\tau) & \text{for} \quad t \geq \tau \end{cases}$

Issue:

Find a stopping time T^* such that the expected delay $E(T - \tau \mid T \geq \tau)$ is minimal given the false alarm probability $P(T < \tau)$.

b) Stopping the Brownian Motion at the Maximum

$B_t,\ 0 \le t \le 1$ standard Brownian motion

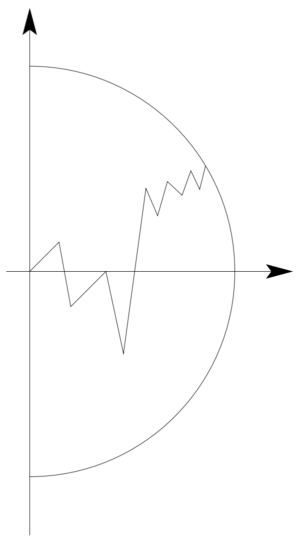
$$R(T) := E\left(B_T - \max_{0\le s\le 1} B_s\right)^2$$

Here T denotes a stopping time of Brownian motion with $0 \le T \le 1$.

Find a stopping time T^* with

$$R(T^*) = \min_T R(T).$$

Idea:



c) Perpetual American Put Option

Samuelson (1965), McKean (1965)

$X_t = \sigma B_t + \mu t$ Brownian Motion with drift μ and variance σ^2.

Find a stopping time T^* which maximizes

$$Ee^{-rT}(K - e^{X_T})^+ 1_{\{T<\infty\}}.$$

Classical Theory

X_t diffusion with infinitesimal generator $\mathcal{D}$

$r(x,t)$ loss at (x,t)

$u(x,t) = \min_{S \geq t} E^{(x,t)} r(X_S, S)$, S stopping time

The optimal continuation region:

$$\mathcal{C}^* = \{(x,t) \mid r(x,t) > u(x,t)\}$$

The optimal stopping rule:

$$T^* = \inf\{t > 0 \mid (X_t, t) \notin \mathcal{C}^*\}.$$

Solution via free boundary problem (FBP):

$$\begin{aligned} u_t &= \mathcal{D}u && \text{in } \mathcal{C}^* \\ u &= r && \text{on } \partial\mathcal{C}^* \\ u_x &= r_x && \text{on } \partial\mathcal{C}^* \end{aligned}$$

See Shiryaev, Optimal Stopping Rules.

A stopping problem corresponds to a stationary FBP, if $\mathcal{D}u = 0$ or $\mathcal{D}u = ru$. The paper discusses the question of the structure of stopping problems with stationary FBP.

Disruption Problem

Dissertation of Shiryaev (1961)

Observations: $W_t = B_t + \theta(t-\tau)^+$ with

$B_t,\ t \geq 0$ standard Brownian motion,

$\theta > 0$ fixed

Filtration: $\mathcal{F}_t = \sigma(W_t;\ 0 \leq s \leq t)$

Change-point: τ random time,

with distribution $\pi = p\delta_0 + (1-p)F$,

where $F(t) = 1 - e^{-\lambda t}$

Risk: $R(T) = P_\pi(T < \tau) + cE_\pi(T-\tau)^+$

Find T^* with $R(T^*) = \min_T R(T)$

Theorem

$T^* = \min\{t > 0 \mid \pi_t \geq p^*\}$ with $\pi_t = P(\tau \leq t \mid \mathcal{F}_t)$

Here p^* is the unique solution of $G(p) = p$ and $G'(p) = -1$, where G is the positive (and finite at 0) solution of

$$\frac{\theta}{2} x^2(1-x^2)G''(x) + \lambda(1-x)G'(x) = -x$$

$$\pi_t = \frac{\varphi_t}{e^{-\lambda t} + \varphi_t} \qquad \text{posterior}$$

where

$$\varphi_t = \frac{p}{1-p} L_t + \int_0^t \frac{L_t}{L_s} \lambda e^{-\lambda s}\, ds$$

and

$$L_t = \exp(\theta W_t - \theta^2 t/2)$$

<u>Case $p = 0$:</u>

$$P(\tau \leq t) = 1 - e^{-\lambda t} \qquad \text{prior}$$

$$R\pi_t = \frac{\pi_t}{1 - e^{-\lambda t}} - 1 \qquad \text{relative posterior}$$

Stopping the Brownian Motion at the Maximum

$B_t,\ 0 \le t \le 1$ standard Brownian motion

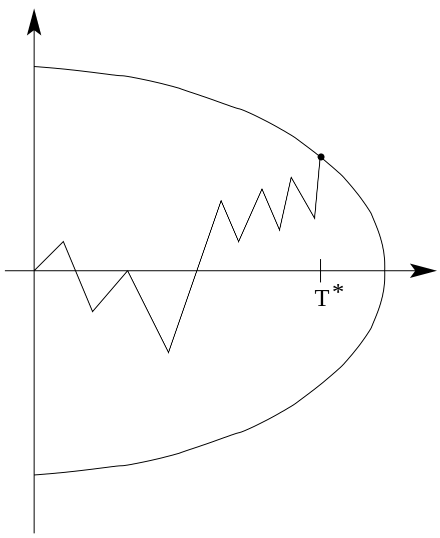
$$R(T) := E\left(B_T - \max_{0\le s\le 1} B_s\right)^2$$

Here T denotes a stopping time of Brownian motion with $0 \le T \le 1$.

Find a stopping time T^* with

$$R(T^*) = \min_T R(T).$$

<u>Idea:</u>



Graversen, Peskir, Shiryaev (2001)

$$T^* = \inf\{t \leq 1 | S_t - B_t \geq z_* \sqrt{1-t}\}$$

where $S_t = \max_{0 \leq s \leq t} B_t$.

z_* is the unique solution of the equation

$$4\Phi(z_*) - 2z_*\varphi(z_*) - 3 = 0, \quad z_* \cong 1,12.$$

Note: $\mathcal{L}(S - B) = \mathcal{L}(|B|)$

It holds:

$$ET^* = \frac{z_*^2}{1 + z_*^2} \cong 0,55, \quad \mathrm{Var}T^* \cong 0,05$$

$$ET^* = EB_{T^*}^2 = z_*^2 E(1 - T^*)$$

Let $V_* = R(T^*)$

Then $V_* = 2 \inf_\sigma E\left(\int_0^\sigma e^{-2s} F\left(|Z_s|\right)\, ds\right) + 1$

where $F(x) = 4\Phi(x) - 3$ and Z_s is a diffusion process with $dZ_t = Z_t\, dt + \sqrt{2}\, d\beta_t$ and $\beta_t,\ t \geq 0$ is a SBM.

Let $W_*(z) = \inf_\sigma E_z\left(\int_0^\sigma e^{-2s} F\left(|Z_s|\right)\, ds\right)$.

Then $V_* = 2W_*(0) + 1$

To determine W_* solve

FBP: $(\mathcal{D} - 2)W(z) = -F(|z|) \quad$ for $-z_* \leq z \leq z_*$
$W(\pm z_*) = 0$
$W'(\pm z_*) = 0$

with $\mathcal{D} = z\dfrac{d}{dz} + \dfrac{d^2}{dz^2}$.

With Ito's formula

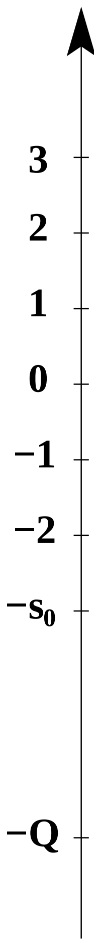
$$E_z\left(e^{-2\tau} W(|Z_\tau|) - W(z)\right) = -E_z \int_0^\tau e^{-2t} F\left(|Z_t|\right)\, dt$$

Find a stopping time σ^* with

$$E_z\left(e^{-2\sigma^*} W\left(|Z_{\sigma^*}|\right)\right) = \max_\tau E\left(e^{-2\tau} W\left(|Z_\tau|\right)\right)$$

Solution: $\sigma^* = \inf\{t \geq 0 \mid |Z_t| = z_*\}$

The Parking Problem



$$S_0 = -Q$$

$$S_n = \sum_{i=1}^{n} X_i - Q$$

X_i i.i.d. geometric (p)

p: probability of empty spot

Park as near as possible at "0"!

Find stopping time T^* of $S_i, i \geq 0$ with

$E \mid S_{T^*} \mid = \min_T E \mid S_T \mid$.

Solution: $T^* = \min\{n \geq 1 \mid S_n \geq -s_0\}$

with $s_0 = \min\{s \in \mathbb{N} \mid 1 - 2(1-p)^s > 0\}$

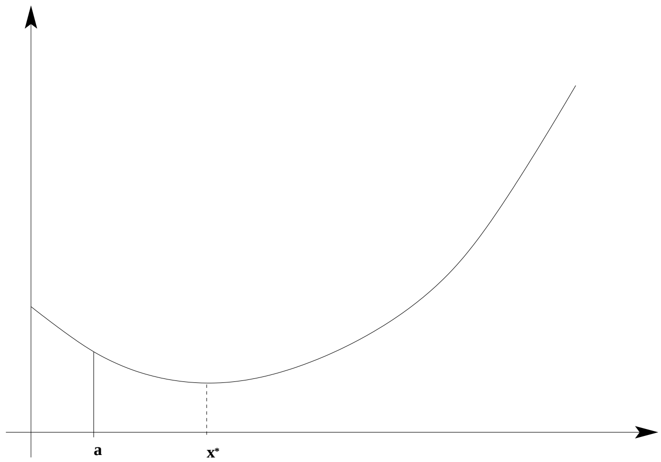
Generalized Parking Problem (GPP)

Let g be a convex function with a unique minimum at $x^* > 0$.

Assume X_i i.i.d. with $EX_i > 0$,

$$S_n = \sum_{i=1}^{n} X_i, \quad S_0 = 0.$$

Find a stopping time T^* with $Eg(S_{T^*}) = \min_T Eg(S_T)$



Solution (Keener, Lerche, Woodroofe '94):

$$T^* = \min\{n \geq 0 \mid S_n \geq a\} \text{ with } a < x^*$$

with $a = \sup\{x \mid H^+ g(x) < g(x)\}$ where H^+ is the ladder-height distribution of $S_n;\ n \geq 1$.

The Main Idea: OS as GPP

Let $(Z_t, \mathcal{F}_t; t \geq 0)$ denote a continuous stochastic process on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$.
Find a stopping time T^* with:

$$E_P\left(Z_{T^*} 1_{\{T^* < \infty\}}\right) = \max_T E_P\left(Z_T 1_{\{T < \infty\}}\right)$$

Idea:
Find a process $(X_t, \mathcal{F}_t; t \geq 0)$, a measure $Q (Q \ll P)$ and a function g with unique maximum at x^* such that

$$Z_t = g(X_t) \left. \frac{dQ}{dP} \right|_{\mathcal{F}_t}$$

Then

$$\begin{aligned} E Z_T 1_{\{T<\infty\}} &= E\left(g(X_T) \left. \frac{dQ}{dP} \right|_{\mathcal{F}_T} \right) \\ &= E_Q\left(g(X_T) 1_{\{T<\infty\}}\right) \\ &\leq g(x^*) Q\left(T < \infty\right) \\ &\leq g(x^*) \end{aligned}$$

With $T^* = \min\{t \geq 0 \mid X_t = x^*\}$ the inequalities become equalities, if $Q\left(T^* < \infty\right) = 1$.

The RN-Densities as Martingales

Let $M_t = \left.\frac{dQ}{dP}\right|_{\mathcal{F}_t}$ where $Q \ll P$.

Then for all $t \geq 0$:

1) $E_P M_t = E_P \left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = E_Q\, 1 = 1$

For $s < t$ it holds:

2) $E\left(M_t \mid \mathcal{F}_s\right) = M_s$, the martingale property.

Of course

3) M_t is nonnegative and $\mathcal{F}_t$-measurable.

Conversely:

Given a stochastic process $M_t;\ t \geq 0$ with properties 1) $-$ 3). It defines a measure Q by

$$Q_t(A) = \int_A M_t\, dP \qquad \text{for } A \in \mathcal{F}_t.$$

The Repeated Significance Test is a Bayes-Test

$W(t), t \geq 0$ Brownian motion with drift θ

Testing sequentially: $H_0 : \theta < 0$ versus $H_1 : \theta > 0$

Prior: $G(d\theta) = \varphi(\tau\theta)\sqrt{\tau}d\theta$

$$R(T,\delta) = \int_{-\infty}^{0} \left(P_\theta\{\delta \text{ rejects } H_0\} + \frac{c}{2}\theta^2 E_\theta T \right) G(d\theta) + \int_{0}^{\infty} \left(P_\theta\{\delta \text{ rejects } H_1\} + \frac{c}{2}\theta^2 E_\theta T \right) G(d\theta)$$

Find (T^*, δ^*) with $R(T^*, \delta^*) = \min_{(T,\delta)} R(T, \delta)$.

$\delta^* = \delta^*_T = 1_{\{W(T)>0\}}$ $\qquad T^* = ?$

PNAS, 83 (1986)

$$
\begin{aligned}
&\int_{-\infty}^{0} P_\theta\{\delta \text{ rejects } H_0\}G(d\theta) + \int_0^\infty \{\delta \text{ rejects } H_1\}G(d\theta) \\
&= \int G_{x,T}(-\infty,0]1_{\{\delta>0\}}Q(dx) \\
&\quad + \int G_{x,T}(0,\infty]1_{\{\delta<0\}}Q(dx) \\
&\geq \int \min_x(G_{x,T}(-\infty,0], G_{x,T}(0,\infty])Q(dx) \\
&= \int \Phi\left(-\frac{|W(T)|}{\sqrt{T+r}}\right) dQ
\end{aligned}
$$

$$
G = N(0, r^{-1}), \qquad Q = \int P_\theta G(d\theta)
$$

$$
\begin{aligned}
&\int \theta^2 E_\theta T G(d\theta) \\
&= \int T \int \theta^2 G_{W(T),T}(d\theta) dQ \\
&= \int T\left(\frac{W(T)^2}{(T+r)^2} + \frac{1}{T+r}\right) dQ \\
&= \int (T+r)\left(\frac{W(T)^2}{(T+r)^2} + \frac{1}{T+r}\right) dQ - 1 \\
&= \int \frac{W(T)^2}{T+r} dQ
\end{aligned}
$$

$$
G_{W(T),T} = N\left(\frac{W(T)}{T+r}, \frac{1}{T+r}\right)
$$

Representation of the risk:

$$R(T,\delta_T^*) = \int g\left(\frac{W(T)^2}{T+r}\right) dQ$$

with $g(x) = \Phi(-\sqrt{x}) + cx/2$

g is convex with unique minimum x^*

$$R(T,\delta_T^*) = \int g\left(\frac{W(T)^2}{T+r}\right) dQ \geq g(x^*)$$

Let $T^* = \min\{t > 0 \mid W(t)^2/(t+r) = x^*\}$

Since $\overline{P}\{T^* < \infty\} = 1$ it follows

$$R(T^*,\delta_{T^*}^*) = g(x^*).$$

The same type of argument holds for the SPRT.

The representation is $R(T,\delta_T^*) = \int g(|\theta|\,|W(T)|)d\overline{Q}$

with $\overline{Q} = \frac{1}{2}(P_\theta + P_{-\theta}), \theta > 0.$

Disruption Problem: The Representation

Let $\pi_t = P(\tau < t \mid \mathcal{F}_t)$.

π_t is a diffusion with

$$d\pi_t = \lambda(1-\pi_t)dt + \theta\pi_t(1-\pi_t)d\overline{W_t}$$

where $\overline{W_t}$ is a standard Brownian motion.

Ito's formula yields:

$$\begin{aligned} dG(\pi_t) &= G'(\pi_t)d\pi_t + \frac{1}{2}G''(\pi_t)(d\pi_t)^2 \\ &= G'(\pi_t)\left[\lambda(1-\pi_t)dt + \theta\pi_t(1-\pi_t)d\overline{W_t}\right] \\ &\quad + \frac{1}{2}G''(\pi_t)\theta^2\pi_t^2(1-\pi_t)^2dt \end{aligned}$$

If G satisfies the equation

$$\frac{\theta^2}{2}x^2(1-x)^2G''(x) + \lambda(1-x)G'(x) = x$$

then

$$G(\pi_t) - G(\pi_0) = \int_0^t \pi_s ds + \int_0^t \theta\pi_s(1-\pi_s)d\overline{W_s}$$

$$\Rightarrow E\left[G(\pi_T) - G(\pi_0)\right] = E\int_0^T \pi_s ds$$

$$\begin{aligned} R(T) &= P(T < \tau) + cE(T - \tau)^+ \\ &= E\left[(1 - \pi_T) + c\int_0^T \pi_s ds\right] \end{aligned}$$

Then with $g(x) = (1 - x) + cG(x)$ yields

$$R(T) = \int g(\pi_T)dP - g(p)$$

g is convex with a unique minimum at p^*.

This insight opens a new direction to Bayes tests of power one for change point problems. Cusum and Mixture stopping rules can be derived as Bayes tests. (Beibel 1996, 1997), (Beibel – Lerche 2003).

Perpetual American Put Option

Samuelson(1965), McKean(1965)

$X_t = \sigma B_t + \mu t$ Brownian Motion with drift μ and variance σ^2.

Find a stopping time T^* which maximizes

$$E_P e^{-rT}(K - e^{X_T})^+ 1_{\{T<\infty\}}.$$

<u>Idea:</u>

Find Q and g with $E_P e^{-rT}(K - e^{X_T})^+ = E_Q g(X_T)$, where $Q \ll P$ and g has a unique maximum at x^*.

Then $T^* = \min\{t \geq 0 \mid X(t) = x^*\}$, if $Q(T^* < \infty) = 1$.

Let

$$f(x) = (K - e^x)^+ \text{ and } M_t = \frac{dQ_t}{dP_t}.$$

Then

$$Ee^{-rT} f(X_T) = Ef(X_T)(e^{X_T})^{-\alpha}(e^{X_T})^{\alpha} e^{-rT}.$$

Choose $g(x) = f(x)e^{-\alpha x}$ and α such that $M_t = e^{\alpha X_t} e^{-rt}$ is a martingale. With

$$\begin{aligned} M_t &= \exp\left[\alpha(\sigma B_t) + \alpha\mu t - rt\right] \\ &= \exp\left[(\alpha\sigma)B_t - t(\alpha\sigma)^2/2\right] \end{aligned}$$

$M_0 = 1$ and M_t is a positive marginale. Then

$$\alpha^{\pm} = -\frac{\mu}{\sigma^2} \pm \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}}.$$

Let $K < 1 + (-\alpha^-)^{-1}$. Then g has a unique maximum at $x^* = \log \frac{\alpha^- K}{\alpha^- - 1} < 0$. Under Q X_t has drift

$$\alpha^- \sigma^2 + \mu = -\sigma^2 \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} < 0.$$

This yields $Q(T^* < \infty) = 1$, if $x^* < 0$.

If $\mu < 0$ then $P(T^* < \infty) = 1$ and

$$\sup_T E_P e^{-rT}(K - e^{X_T})^+ = E_P e^{-rT^*}(K - e^{X_{T^*}})^+$$

One-Sided Boundaries

Let h be measurable, $X_t = \sigma B_t + \mu t$ Brownian motion with drift μ and variance σ^2. Find a stopping time T^* which maximizes

$$Ee^{-rT}h(X_T)1_{\{T<\infty\}}.$$

Let $\alpha_{1,2} = -\frac{\mu}{\sigma^2} \pm \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}}$. Then $M_t^{(i)} = e^{-rt}e^{\alpha_i X_t}$, $i = 1, 2$ are positive martingales.

<u>Theorem 1:</u>

If $0 < C_1 = \sup_{x\in\mathbb{R}}(e^{-\alpha_1 x}h(x)) < \infty$ and $C_1 = e^{-\alpha_1 x_1}h(x)$ for some $x_1 > 0$, then

$$\sup_T Ee^{-rT}h(x_T)1_{\{T<\infty\}} = C_1,$$

$$T^* = \inf\{t > 0 \mid X_t = x_1\}.$$

<u>Theorem 2:</u>

If $0 < C_2 = \sup_{x\in\mathbb{R}}(e^{-\alpha_2 x}h(x)) < \infty$ and $C_2 = e^{-\alpha_2 x_2}h(x_2)$ for some $x_2 > 0$, then

$$\sup_T Ee^{-rT}h(x_T)1_{\{T<\infty\}} = C_2,$$

$$T^* = \inf\{t > 0 \mid X_t = x_2\}.$$

Two-Sided Boundaries

Let $h(x)$ be nonnegative and measurable with

a) $\sup_{x\leq 0}(e^{-\alpha_1 x}h(x)) > \sup_{x\geq 0}(e^{-\alpha_1 x}h(x)) > 0$ and

b) $\sup_{x\geq 0}(e^{-\alpha_2 x}h(x)) > \sup_{x\leq 0}(e^{-\alpha_2 x}h(x)) > 0$.

Examples:

1.) $h(x) = x^2$

2.) $h(x) = \max\{(L - e^x)^+, (e^{-x} - K)^+\}$

Let $p \in [0,1]$. Let $M_t = pM_t^{(1)} + (1-p)M_t^{(2)}$. Then

$$Ee^{-rT}h(X_t) = EM_T \frac{h(X_T)}{pe^{\alpha_1 X_T} + (1-p)e^{\alpha_2 X_T}}.$$

Lemma: If a) and b) holds, there exists a $p^* \in (0,1)$ with $\sup_{x\geq 0} G_{p^*}(x) = \sup_{x\leq 0} G_{p^*}(x)$, where

$$G_p(x) = \frac{h(x)}{pe^{\alpha_1 x} + (1-p)e^{\alpha_2 x}}.$$

Theorem 3:

Let $C^* = \sup_{x\in \mathbb{R}} G_{p^*}(x)$. If there exists points $x_1 > 0$ and $x_2 < 0$ with $G_{p^*}(x_1) = C^* = G_{p^*}(x_2)$, then

$$\sup_T Ee^{-rT}h(X_T)1_{\{T<\infty\}} = C^*$$

and

$$T^* = \inf\{t > 0 \mid X_t = x_1 \text{ or } X_t = x_2\}.$$

Stopping of Diffusions with Random Exponential Discounting

$B_t, t \geq 0 \quad SBM$

X_t diffusion with $X_0 = x$ and

$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$

$h : \mathbb{R} \to \mathbb{R}_+$ a continuous function

Find a stopping time T^* of X with

$$E\Big(e^{-A(T)}h(X_T)1_{\{T<\infty\}}\Big) = \max$$

$A(s)$: additive continuous stochastic process adapted to $\mathcal{F}^X$

$$A(s+t) = A(s) + A(t) \circ \theta_s$$

Example: $r(x) \geq 0, \alpha > 0$

$$E\left(\exp\left\{-\int_0^T r(B_t)dt\right\}(B_T)^\alpha 1_{\{T<\infty\}}\right) = \max$$

$$X(0) = x_0,\ x_0 \in I$$

Assume $\psi_+(x_0) = \psi_-(x_0) = 1$

Then

$$e^{-A_t} h(X_t) = M_t \frac{h(X_t)}{p\psi_+(X_t) + (1-p)\psi_-(X_t)}$$

with $M_t = e^{-A_t} \left(p\psi_+(X_t) + (1-p)\psi_-(X_t) \right)$

for any $p \in [0,1]$ and $0 \le t < \infty$.

M_t is a positive local martingale and hence

$$E(M_T 1_{\{T<\infty\}}) \le 1.$$

<u>Problem:</u>

Maximize $\frac{h(x)}{p\psi_+(x)+(1-p)\psi_-(x)}$ over all $x \in I$ with a proper p.

How to choose the martingales?

$$\psi_+(x) \;=\; \begin{cases} E_x\left(e^{-A(T_{x_0})}1_{\{T_{x_0}<\infty\}}\right) & \text{for } x \le x_0 \\ \left[E_{x_0}\left(e^{-A(T_x)}1_{\{T_x<\infty\}}\right)\right]^{-1} & \text{for } x \ge x_0 \end{cases}$$

$$\psi_-(x) \;=\; \begin{cases} \left[E_{x_0}\left(e^{-A(T_x)}1_{\{T_x<\infty\}}\right)\right]^{-1} & \text{for } x \le x_0 \\ E_x\left(e^{-A(T_{x_0})}1_{\{T_{x_0}<\infty\}}\right) & \text{for } x \ge x_0 \end{cases}$$

$$\begin{aligned} M_t^{(+)} &= e^{-A(t)}\psi_+(X_t) \\ M_t^{(-)} &= e^{-A(t)}\psi_-(X_t) \end{aligned} \quad \text{are u.i. martingales with}$$

$$E_x(M_{T_b}^{(+)}1_{\{T_b<\infty\}}) = \psi_+(x) \quad \text{for } b \ge x \text{ on } 0 \le t \le T_b$$

$$E_x(M_{T_a}^{(-)}1_{\{T_a<\infty\}}) = \psi_-(x) \quad \text{for } x \ge a \text{ on } 0 \le t \le T_a.$$

<u>Note:</u>

If $A(t) = \int_0^t r(X_s)ds$ with $r(x) \ge 0$, then $\psi_\pm(x)$ are the solutions of $\mathcal{D}\psi = r \cdot \psi$ with appropriate boundary conditions.

$$\mathcal{D} = \mu(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma(x)\frac{\partial^2}{\partial x^2}.$$

Distinguish the following Cases

1) $$\sup_{x \geq x_0, x \in I} (h(x)/\psi_+(x)) = \infty$$

2) $$\sup_{x \leq x_0, x \in I} (h(x)/\psi_-(x)) = \infty$$

3) $$0 < C^* = \sup_{x \in I} \frac{h(x)}{\psi_+(x)} = \sup_{x \geq x_0} \frac{h(x)}{\psi_+(x)}$$

4) $$0 < C^* = \sup_{x \in I} \frac{h(x)}{\psi_-(x)} = \sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_-(x)}$$

5) $$0 < \sup_{x \geq x_0, x \in I} (h(x)/\psi_+(x)) < \infty$$
$$0 < \sup_{x \leq x_0, x \in I} (h(x)/\psi_-(x)) < \infty$$
and
$$\sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_+(x)} > \sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_+(x)} \text{ and}$$
$$\sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_-(x)} > \sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_-(x)}$$

In case 5) there exists a $p^* \in (0,1)$ such that

$$\sup_{x \geq x_0, x \in I} \frac{h(x)}{p^*\psi_+(x) + (1-p^*)\psi_-(x)} = \sup_{x \leq x_0, x \in I} \frac{h(x)}{p^*\psi_+(x) + (1-p^*)\psi_-(x)}.$$

Case 3

<u>Theorem "3":</u>

If $0 < C^* = \sup_{x \in I} \frac{h(x)}{\psi_+(x)} = \sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_+(x)} < \infty$

Then

$$\sup_T E_{x_0} \left\{ e^{-A_T} h(X_T) 1_{\{T<\infty\}} \right\} = C^* \qquad (+)$$

If there exists a point $x^* \geq x_0$ with $C^* = h(x^*)/\psi_+(x^*)$, then the supremum in (+) is attained by

$$T^* = \inf \{ t \geq 0 \mid X_t = x^* \} .$$

This holds since

$$E_{x_0} e^{-A_{T^*}} h(X_{T^*}) = E_{x_0} \frac{h(X_{T^*})}{\psi_+(x)} M_{T^*} = C^*.$$

Case 5

Theorem "5":

Let p^* be such that

$$0 < \sup_{x \geq x_0, x \in I} \frac{h(x)}{p^*\psi_+(x) + (1-p^*)\psi_-(x)} = \sup_{x \leq x_0, x \in I} \frac{h(x)}{p^*\psi_+(x) + (1-p^*)\psi_+(x)}$$

then $\sup_T E_{x_0}\left(e^{-A_T}h(X_T)1_{\{T<\alpha\}}\right) = C^*$.

If there exist points $x_1 > x_0$ and $x_2 < x_0$ such that

$$\frac{h(x_1)}{p^*\psi_+(x) + (1-p^*)\psi_-(x)} = \frac{h(x_2)}{p^*\psi_+(x) + (1-p^*)\psi_-(x)} = C^*,$$

then the supremum is attained for

$T^* = \inf\{t > 0 \mid X_t = x_1, X_2 = x_2\}$.

Generalized Parking Problem: Discrete Case, Details

$X_1, X_2, \ldots$ i.i.d. with $EX_i > 0$,

$$S_n = \sum_{i=1}^{n} X_i, \quad S_0 = 0.$$

Find stopping time T^* with $Eg(S_{T^*}) = \min_T Eg(S_T)$

Solution: $T^* = \min\{n \geq 0 \mid S_n \geq a\}$
with

$$a = \sup\{x \mid H^+g(x) < g(x)\}$$

$$H^+g(x) := \int g(x+y) H^+(dy)$$

$$H^+(y) := P(S_\eta \leq y)$$

$$\eta := \min\{n > 0 \mid S_n > 0\}$$

H^+: the distribution of the first ladder height S_η.

Let $K(x) = \int_0^x \frac{1 - H^+(y)}{\gamma_1} \, dy$

with $\gamma_i = \int y^i \, dH^+(y)$, $i \in \mathbb{N}$.

Theorem

If $Kg(x) < \infty$ for all $0 \leq x < \infty$, then $Kg(x)$ is minimized at $x = a$.

Example 1:

If $g(x) = |x - b| \, \forall \, x \in \mathbb{R} \Rightarrow b - a = \text{med}(K)$

Example 2:

If $g(x) = (x - b)^2 \, \forall \, x \in \mathbb{R}$

$\Rightarrow b - a = \text{mean}(K) = \gamma_2/\gamma_1$

Example 3:

If $g(x) = e^{-x} + cx \, \forall \, x \in \mathbb{R}, \; 0 < c < 1$

$\Rightarrow b = \log(1/c)$.

If $\int x^2 H^+(dx) < \infty$ and if $\kappa := \int_0^\infty e^{-x} K(dx)$

$\Rightarrow Kg(x) = \kappa e^{-x} + c(x + \frac{\gamma_2}{2\gamma_1})$ and is minimized when $x = \log(\kappa/c)$.

Related Literature

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