A Note on Optimal Stopping of Regular Diffusions under Random Discounting

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Summary. Let X be a one-dimensional regular diffusion, A a positive continuous additive functional of X, and h a measurable real-valued function. A method is proposed to determine a stopping rule T^* that maximizes $E\{e^{-A_T}h(X_T)1_{\{T<\infty\}}\}$ over all stopping times T of X. Several examples, some related to Mathematical Finance, are discussed.

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1 Introduction

Let B denote standard Brownian motion and let r be a nonnegative measurable function on the real numbers $\mathbf{R}$. Let us consider the problem of maximizing the gain

$$E\left(e^{-\int_0^T r(B_s)ds}(B_T^+)^\alpha 1_{\{T<\infty\}}\right)$$

for some $\alpha > 0$ over all stopping times T of B. To find an optimal stopping time for this and related problems we extend the approach of Beibel and Lerche (1997). In the present case that means to represent the payoff $\exp\{-\int_0^t r(B_s)ds\}(B_t^+)^\alpha$ as

$$e^{-\int_0^t r(B_s)ds}\psi(B_t)\frac{(B_t^+)^\alpha}{\psi(B_t)},$$

where ψ is choosen such that $\exp\{-\int_0^t r(B_s)ds\}\psi(B_t)$ becomes a positive local martingale and $(x^+)^\alpha/\psi(x)$ is a function which assumes its maximum over $\mathbf{R}$ at some point $x^* > 0$. Then we obtain for all stopping times T of B that

$$\begin{aligned} E\left(e^{-\int_0^T r(B_s)ds}(B_T^+)^\alpha 1_{\{T<\infty\}}\right) &\leq \left(\sup_{x\in\mathbf{R}}\frac{(x^+)^\alpha}{\psi(x)}\right) E\left(e^{-\int_0^T r(B_s)ds}\psi(B_T) 1_{\{T<\infty\}}\right) \\ &\leq \psi(0)\frac{(x^*)^\alpha}{\psi(x^*)}. \end{aligned}$$

In order to prove the optimality of $T^* = \inf\{t \geq 0 | B_t = x^*\}$, it is left to show that

$$E\left(e^{-\int_0^{T^*} r(B_s)ds}\psi(B_{T^*}) 1_{\{T^*<\infty\}}\right) = \psi(0). \tag{1}$$

To be more explicit, let us consider the case $r(x) = r1_{[0,\infty)}(x)$ for some positive constant r. Let

$$\psi(x) = \begin{cases} (1/2)\Big(\exp(\sqrt{2r}x) + \exp(-\sqrt{2r}x)\Big) & \text{for} \quad x \geq 0 \\ 1 & \text{for} \quad x < 0. \end{cases} \tag{2}$$

Then it holds that $\sup_{x\in\mathbf{R}}[(x^+)^\alpha/\psi(x)] < \infty$ and the supremum is attained at the unique positive solution x^* of the implicit equation

$$\sqrt{2r}x\left(e^{\sqrt{2r}x} - e^{-\sqrt{2r}x}\right) = \alpha\left(e^{\sqrt{2r}x} + e^{-\sqrt{2r}x}\right).$$

The generalized Itô formula (see Rogers and Williams (1987), IV.45.2) yields that $\exp\{-r\int_0^t 1_{(0,\infty)}(B_s)ds\}\psi(B_t)$ is indeed a positive local martingale. Moreover ψ is nondecreasing. Therefore the process $\exp\{-\int_0^t r(B_s)ds\}\psi(B_t)$ is bounded by $\psi(x^*)$ for $0 \leq t \leq T^*$. Since $P(T^* < \infty) = 1$, this yields (1). Thus T^* is an optimal stopping rule.

The approach just presented can be applied to a general class of optimal stopping problems. Let X denote a one-dimensional regular diffusion in the sense of Feller, Itô, and McKean on some probability space $(\Omega, \mathcal{F})$. 'Regular' here means that $P(T_y < \infty | X_0 = x) > 0$ holds for all $y \in I$ and $x \in \text{int}(I)$, with $T_y = \inf\{t \geq 0 | X_t = y\}$. In the sequel we assume that the structure of $(\Omega, \mathcal{F})$ is such that there exist measurable shift operators θ_s for $s \geq 0$ with $X_{t+s}(\omega) = X_t(\theta_s(\omega))$ for all $\omega \in \Omega$ and $s, t \geq 0$. Let $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$. Let h be a measurable function and r be a positive constant. Let $(A_s; s \geq 0)$ be a nonnegative continuous additive functional of X, which means that A satisfies the following conditions:

i) $A_t \in [0, \infty)$ for $0 \leq t < \infty$.

ii) A_t is measurable with respect to $\mathcal{F}_t^X$.

iii) $A_{s+t} = A_s + A_t \circ \theta_s$.

iv) A_t is continuous in t.

We will consider the following general optimal stopping problem.

Problem 1 *Find a stopping time T^* of X that maximizes*

$$E\left(e^{-A_T}h(X_T)1_{\{T<\infty\}}\Big|X_0 = x\right)$$

among all stopping times T of X.

Note that the stopping time $T \equiv +\infty$ yields the expected payoff 0. Hence we will never stop whenever $h(X_t) < 0$. This means we can always replace h by $\max\{h, 0\}$.

The case of deterministic discounting $A_t = rt$ is treated by Mucci (1978) and Salminen (1985). Their methods and conditions on h differ somewhat from ours. Under additional technical conditions on the drift and diffusion coefficient of X, Bensoussans and Lions (1982) treat the case where h is bounded and $A_t = \int_0^t r(X_s)ds$ with a positive bounded measurable function r with $r \geq \beta$ for some strictly positive constant β (see Bensoussans and Lions (1982), Theorem 3.19, p. 387).

In the following the state space of X is an interval I on the real line. The interior of I will be denoted by $\text{int}(I)$. An endpoint z of I with $z \in I$ will be called closed endpoint. We will sometimes write P_x and E_x instead of $P(.|X_0 = x)$ and $E(.|X_0 = x)$ respectively. For later use we introduce the following classification of closed boundary points. A closed endpoint z of I is called a reflecting boundary point if for some $y \in \text{int}(I)$ (and hence for all $y \in I$) it holds that $P(T_y < \infty|X_0 = z) > 0$ and it is called an absorbing boundary point if $P(T_y < \infty|X_0 = z) = 0$ holds for all $y \in \text{int}(I)$.

This paper has the following structure. In Section 2 we use the approach of Itô and McKean (1965, section 4.6) and construct suitable functions ψ for which

$e^{-A_t}\psi(X_t)$ are local martingales. Section 3 contains our main results. In Section 4 we present further examples. The proofs of the results in Section 3 are given in Section 5 and Section 6 covers briefly some extensions.

2 The functions ψ_+ and ψ_-

In this section we follow the ideas of section 4.6 of Itô and McKean (1965). The proofs are similar to the arguments there and in section V.46 of Rogers and Williams (1987). We therefore omit them. Let x_0 denote some point in $\mathrm{int}(I)$. We then define the functions ψ_+ and ψ_- on $\mathrm{int}(I)$ by [compare Itô and McKean (1965), p. 128-129, equations 2a) and 2b)]

$$\psi_+(x) = \begin{cases} E_x\left(e^{-A(T_{x_0})}1_{\{T_{x_0}<\infty\}}\right) & \text{for} \quad x \leq x_0 \\ 1/E_{x_0}\left(e^{-A(T_x)}1_{\{T_x<\infty\}}\right) & \text{for} \quad x \geq x_0 \end{cases} \tag{3}$$

and

$$\psi_-(x) = \begin{cases} 1/E_{x_0}\left(e^{-A(T_x)}1_{\{T_x<\infty\}}\right) & \text{for} \quad x \leq x_0 \\ E_x\left(e^{-A(T_{x_0})}1_{\{T_{x_0}<\infty\}}\right) & \text{for} \quad x \geq x_0 \end{cases} \tag{4}$$

The values at closed endpoints of I can be computed by using the continuity properties of ψ_+ and ψ_-. Let

$$M_t^{(+)} = e^{-A(t)}\psi_+(X_t) \quad \text{and} \quad M_t^{(-)} = e^{-A(t)}\psi_-(X_t).$$

Lemma 1 *$M_t^{(+)}$ is a uniformly integrable P_x-martingale on $0 \leq t \leq T_b$ for all $b \geq x$ and $M_t^{(-)}$ is a uniformly integrable P_x-martingale on $0 \leq t \leq T_a$ for all $a \leq x$. For any $x \leq b$ we have*

$$E_x\left(M_{T_b}^{(+)}1_{\{T_b<\infty\}}\right) = \psi_+(x)$$

and for any $x \geq a$ we have

$$E_x\left(M_{T_a}^{(-)}1_{\{T_a<\infty\}}\right) = \psi_-(x)\ .$$

Clearly $\psi_+(a) > 0$ if a is a reflecting left boundary point and $\psi_+(a) = 0$ if a is an absorbing left boundary point. Also $\psi_-(b) > 0$ if b is a reflecting right boundary point and $\psi_-(b) = 0$ if b is an absorbing right boundary point. Moreover $\psi_+(b) > 0$ for any closed right endpoint b and $\psi_-(a) > 0$ for any closed left endpoint. Moreover:

Lemma 2 *The functions ψ_+ and ψ_- are continuous and strictly positive on* $\mathrm{int}(I)$*. If a is a closed left endpoint of I, then*

$$\lim_{x \downarrow a} \psi_+(x) = \psi_+(a) \quad \textit{and} \quad \lim_{x \downarrow a} \psi_-(x) = \psi_-(a) \ .$$

If b is a closed right endpoint of I, then

$$\lim_{x \uparrow b} \psi_+(x) = \psi_+(b) \quad \textit{and} \quad \lim_{x \uparrow b} \psi_-(x) = \psi_-(b) \ .$$

The function ψ_+ is nondecreasing on I and ψ_- is nonincreasing on I.

When $A_t = \int_0^t r(X_s)ds$ is given with some nonnegative measurable function r on I, we may use the Feynman–Kac formula to determine the functions ψ_+ and ψ_- (see von Weizsäcker and Winkler (1990), section 12.3). When $\mathcal{G}$ denotes the infinitesimal operator of X, then one has to solve the equation $\mathcal{G}\psi(x) = r(x)\psi(x)$ subject to the appropriate boundary conditions.

3 Optimal Stopping

Let us first assume that for all $x \in I$ holds $\psi_+(x) > 0$ and $\psi_-(x) > 0$. This means that we have no absorbing boundary points. We will extend our results to absorbing boundaries in Section 6. Theorem 1 below corresponds to the case where the optimal payoff is infinite. Theorem 2 essentially covers the case where the optimal stopping rule is 'one-sided' whereas Theorem 3 treats the 'two-sided' case.

We fix the starting point X_0 of the process X at some point $x_0 \in I$. Without loss of generality we may assume that the functions ψ_+ and ψ_- are standardized in such a way that $\psi_+(x_0) = \psi_-(x_0) = 1$. We may write the payoff as

$$e^{-A_t} h(X_t) = e^{-A_t} \Big(p\psi_+(X_t) + (1-p)\psi_-(X_t)\Big) \frac{h(X_t)}{p\psi_+(X_t) + (1-p)\psi_-(X_t)} \tag{5}$$

for any $p \in [0,1]$ and $0 \le t < \infty$. The process $e^{-A_t} \left(p\psi_+(X_t) + (1-p)\psi_-(X_t)\right)$ is a positive local martingale and hence a supermartingale. This yields for any stopping time T of X

$$E_{x_0} \left(e^{-A_T} \left(p\psi_+(X_T) + (1-p)\psi_-(X_T)\right) 1_{\{T<\infty\}} \right) \;\le\; 1. \tag{6}$$

The problem of maximizing

$$E_{x_0} \left(e^{-A_T} h(X_T) 1_{\{T<\infty\}} \right)$$

over all stopping times T is equivalent to the problem of maximizing

$$\frac{h(x)}{p\psi_+(x) + (1-p)\psi_-(x)}$$

over all $x \in I$ for a proper choice of p. We have to consider five different cases. It is easy to see that our cases cover all possible functions h. However, they are not exclusive.

i) $\sup_{x \ge x_0, x \in I} [h(x)/\psi_+(x)] = +\infty$. Then $p = 1$ is a proper choice and Theorem 1 applies.

ii) $\sup_{x \le x_0, x \in I} [h(x)/\psi_-(x)] = +\infty$. Then $p = 0$ is a proper choice and Theorem 1 applies.

iii) $0 < C^* = \sup_{x \in I}[h(x)/\psi_+(x)] = \sup_{x \geq x_0, x \in I}[h(x)/\psi_+(x)] < \infty$. Then $p = 1$ is a proper choice and Theorem 2 i) applies.

iv) $0 < C^* = \sup_{x \in I}[h(x)/\psi_-(x)] = \sup_{x \leq x_0, x \in I}[h(x)/\psi_-(x)] < \infty$. Then $p = 0$ is a proper choice and Theorem 2 ii) applies.

v)

$$0 < \sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_+(x)} < \infty \quad \text{and} \quad 0 < \sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_-(x)} < \infty,$$

and at the same time

$$\sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_+(x)} > \sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_+(x)} \quad \text{and} \quad \sup_{x \geq x_0, x \in I} \frac{h(x)}{\psi_-(x)} > \sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_-(x)}.$$

Similar arguments as in Beibel and Lerche (1997) then provide the existence of a $p^* \in (0, 1)$ such that

$$\sup_{x \geq x_0, x \in I} \frac{h(x)}{p^*\psi_+(x) + (1 - p^*)\psi_-(x)} = \sup_{x \leq x_0, x \in I} \frac{h(x)}{p^*\psi_+(x) + (1 - p^*)\psi_-(x)} . \tag{7}$$

Now $p = p^*$ is a proper choice and Theorem 3 applies.

The following elementary example illustrates all possibilities. Let X denote the three-dimensional Bessel process and put $X_0 = 1$. We have for $x, y \in (0, \infty)$ that $P_x(T_y < \infty) = \min\{(y/x), 1\}$. Let $A \equiv 0$. Then $\psi_+(x) = 1$ and $\psi_-(x) = 1/x$.

The cases i) and iii) are trivial.

case ii) If $h_2(x) = 1/x^2$, then $\sup_{0<x\leq 1}[h_2(x)/\psi_-(x)] = +\infty$.

case iv) If $h_4(x) = (1-x)^+$, then $\sup_{x>0}[h_4(x)/\psi_-(x)] = \sup_{0<x\leq 1}[h_4(x)/\psi_-(x)] = 1/4$. The supremum is assumed at $x = 1/2$.

case v) If

$$h_5(x) = \begin{cases} 1 - x^2 & \text{for} \quad 0 < x \leq 1 \\ 1/x - 1/x^3 & \text{for} \quad 1 < x < \infty \end{cases} ,$$

then

$$0 < \sup_{x \geq 1} \frac{h_5(x)}{\psi_+(x)} = \frac{2}{3\sqrt{3}} = \sup_{0<x\leq 1} \frac{h_5(x)}{\psi_-(x)},$$

and at the same time

$$\sup_{0<x\leq 1} \frac{h_5(x)}{\psi_+(x)} = 1 = \sup_{x\geq 1} \frac{h_5(x)}{\psi_-(x)}.$$

Moreover $xh_5(x)/[1+x] = x(x-1)$ for $x \in (0,1]$ and $xh_5(x)/[1+x] = (1/x)(1-1/x)$ for $x \in (1,\infty)$. Therefore

$$\sup_{x\geq 1} \frac{h_5(x)}{0.5\psi_+(x) + 0.5\psi_-(x)} = 0.5 = \sup_{0<x\leq 1} \frac{h_5(x)}{0.5\psi_+(x) + 0.5\psi_-(x)} \ .$$

So, $p^* = 0.5$. Note that the supremum is attained at $x = 0.5$ and $x = 2$ respectively.

Theorem 1 *If*

$$\sup_{x\geq x_0, x\in I} \frac{h(x)}{\psi_+(x)} = +\infty \quad or \quad \sup_{x\leq x_0, x\in I} \frac{h(x)}{\psi_-(x)} = +\infty,$$

then

$$\sup_T E_{x_0}\left(e^{-A_T} h(X_T) 1_{\{T<\infty\}}\right) = +\infty.$$

Theorem 2 *i) If*

$$0 < C^* = \sup_{x\in I} \frac{h(x)}{\psi_+(x)} = \sup_{x\geq x_0, x\in I} \frac{h(x)}{\psi_+(x)} < \infty,$$

then

$$\sup_T E_{x_0}\left\{e^{-rT} h(X_T) 1_{\{T<\infty\}}\right\} = C^* \ . \tag{8}$$

If there exists a point $x^* \geq x_0$ *with* $C^* = h(x^*)/\psi_+(x^*)$*, then the supremum in (8) is attained for* T^* *with* $T^* = \inf\{t \geq 0|\ X_t = x^*\}$.

ii) If

$$0 < C^* = \sup_{x \in I} \frac{h(x)}{\psi_-(x)} = \sup_{x \leq x_0, x \in I} \frac{h(x)}{\psi_-(x)} < \infty \ ,$$

then

$$\sup_T E_{x_0} \left\{ e^{-rT} h(X_T) 1_{\{T<\infty\}} \right\} = C^* \ . \tag{9}$$

If there exists a point $x^* \leq x_0$ *with* $C^* = h(x^*)/\psi_-(x^*)$*, then the supremum in (9) is attained for* T^* *with* $T^* = \inf\{t \geq 0|\ X_t = x^*\}$.

Theorem 3 *Let* p^* *be such that*

$$\begin{aligned} 0 &< \sup_{x \geq x_0, x \in I} \frac{h(x)}{p^* \psi_+(x) + (1-p^*)\psi_-(x)} \\ &= \sup_{x \leq x_0, x \in I} \frac{h(x)}{p^* \psi_+(x) + (1-p^*)\psi_-(x)} = C^* < \infty \ , \end{aligned}$$

then

$$\sup_T \left\{ E_{x_0} e^{-rT} h(X_T) 1_{\{T<\infty\}} \right\} = C^* \ . \tag{10}$$

If there exist points $x_1 > x_0$ *and* $x_2 < x_0$ *such that*

$$\frac{h(x_1)}{p^* \psi_+(x_1) + (1-p^*)\psi_-(x_1)} = \frac{h(x_2)}{p^* \psi_+(x_2) + (1-p^*)\psi_-(x_2)} = C^* \ ,$$

then the supremum in (10) is attained for

$$T^* = \inf\{t > 0 | X_t = x_1 \ \mathit{or}\ X_t = x_2\} \ .$$

4 Further Examples

Let B denote standard Brownian motion throughout this section.

4.1 Standard Brownian Motion — State-dependent Discounting

Put $X = B$ and $x_0 = 0$. Let $r(.)$ be a nonnegative measurable function on $\mathbf{R}$. The function $\phi(x,y) = E_x e^{-\int_0^{T_y} r(X_s)ds}$ is closely related to the Feynman–Kac functional $\exp(-\int_0^t r(X_s)ds)$ and the stationary Schrödinger equation $\frac{1}{2}\psi'' = r\psi$.

a) We have already discussed the particular example $r(x) = r1_{[0,\infty)}(x)$ for some positive constant r. The solution we have presented in the introduction can be viewed as a special case of our general theorems.

b) Let $r(x) = rx^2$ for some constant $r > 0$. Put

$$\psi(x) = e^{-\frac{x^2}{4}} \frac{2^{\frac{5}{4}}}{\Gamma(1/2)} \int_0^{+\infty} e^{xt-\frac{t^2}{2}} \frac{1}{\sqrt{t}} dt \ .$$

In the notation of Magnus and Oberhettinger (1954) (see pp. 91-94) $\psi(x)$ equals $2^{\frac{5}{4}} D_{-1/2}(-x)$; the function $D_{-1/2}$ is usually called Weber function (see also Shepp (1982), Section 3). The function ψ solves $\psi'' = (1/4)x^2\psi$ and we have $\psi(0) = 1$. Hence $\psi(\sqrt{8/r}x)$ is a solution of $(1/2)\psi'' = rx^2\psi$. We have $\psi \geq 0$ and $\lim_{x\to-\infty} \psi(x) = 0$. This yields that ψ is bounded on the interval $(-\infty, y]$ for any $x \in \mathbf{R}$. If we apply the optional stopping theorem to the local martingale $e^{-r\int_0^t X_s^2 ds}\psi(\sqrt{8/r}X_t)$, we obtain for $y \geq 0$

$$E_0 e^{-r\int_0^{T_y} X_s^2 ds} = \frac{1}{\psi\left(\sqrt{\frac{8}{r}}y\right)} \quad \text{and} \quad E_x e^{-r\int_0^{T_0} X_s^2 ds} = \psi\left(\sqrt{\frac{8}{r}}x\right)$$

for $x \leq 0$. According to (4) this provides $\psi_+(x) = \psi((8/r)^{1/2}x)$. Suppose now we are interested in maximizing

$$E_0\left(e^{-r\int_0^T X_s^2 ds}\left(X_T^+\right)^\alpha 1_{\{T<\infty\}}\right)$$

for some $\alpha > 0$. We have $\lim_{x\to+\infty}[\sqrt{x}\psi(x)/e^{\frac{x^2}{4}}] = K$ for some strictly positive constant K. This provides $\sup_{x\in\mathbf{R}}[(x^+)^\alpha/\psi_+(x)] = \sup_{x\geq 0}[(x^+)^\alpha/\psi_+(x)] < \infty$. Theorem 2 i) yields that $T^* = \inf\{t \geq 0 | X_t = x^*\}$, where x^* is the positive solution of the transzendental equation $\alpha\psi_+(x) = x\psi_+'(x)$.

4.2 Standard Brownian Motion — Discounting With Local Time

Put $X = B$ and $x_0 = 0$. Let L denote the local time of B at zero. We will now discuss the problem of maximizing $E_0\left(e^{-rL_T}(X_T^+)^\alpha 1_{\{T<\infty\}}\right)$ for some $\alpha, r > 0$. Let

$$\psi(x) = \begin{cases} 1 + rx & \text{for} \quad x \geq 0 \\ 1 & \text{for} \quad x < 0 \end{cases}$$

The generalized Itô formula (see Rogers and Williams (1987), IV.45.2) yields $d[e^{-rL_t}\psi(X_t)] = re^{-rL_t}dX_t$. Therefore $e^{-rL_t}(1 + rX_t^+)$ is a positive local martingale. We have $\left|e^{-rL_t}(1 + rX_t^+)\right| \leq 1 + rx$ on $0 \leq t \leq T_x$ for any $x \geq 0$ and hence obtain $E_0 \exp\{-rL_{T_x}\} = 1/(1 + rx)$ for $x \geq 0$. For $x < 0$ it holds that $P_x(L_{T_0} = 0) = 1$. This yields together $\psi_+(x) = \psi(x)$.

a) If $\alpha > 1$, we obtain $\sup_{x\geq 0}[(x^+)^\alpha/\psi_+(x)] = +\infty$ and so Theorem 1 yields

$$\sup_T E_0\left(e^{-rL_T}(X_T^+)^\alpha 1_{\{T<\infty\}}\right) = +\infty$$

b) If $\alpha = 1$, we obtain $\sup_{x\geq 0}[(x^+)^\alpha/\psi_+(x)] = 1/r$ and therefore

$$\sup_T E_0\left(e^{-rL_T}(X_T^+)^\alpha 1_{\{T<\infty\}}\right) = \frac{1}{r}\ .$$

The function $x/(1+rx)$ does not attain its supremum over $[0,+\infty)$ at some finite point and so there exists no optimal stopping rule.

c) If $0<\alpha<1$, we obtain

$$\sup_{x\geq 0}\frac{(x^+)^\alpha}{\psi_+(x)}=(1-\alpha)\left(\frac{\alpha}{1-\alpha}\right)^\alpha\left(\frac{1}{r}\right)^\alpha=C^*$$

and this supremum is attained at $x^*=\alpha/[r(1-\alpha)]$. Theorem 2 i) implies

$$\sup_T E_0\left(e^{-rL_T}(X_T^+)^\alpha 1_{\{T<\infty\}}\right)=C^*=E_0\left(e^{-rL_{T^*}}(X_{T^*}^+)^\alpha 1_{\{T^*<\infty\}}\right)$$

where $T^*=\inf\{t>0|X_t=\alpha/[(1-\alpha)r]\}$.

4.3 Russian Options

Let $\mu\in\mathbf{R}$ and $\sigma>0$. Let

$$S_t=\exp\left\{\sigma B_t+\left(\mu-\frac{\sigma^2}{2}\right)t\right\}$$

and $M_t=\max_{0\leq s\leq t}S_s$. Then $dS_t=\mu S_t dt+\sigma S_t dB_t$. Let $r>\mu$. We now consider the problem of maximizing $E(e^{-rT}M_T 1_{\{T<\infty\}})$ over all stopping times T of S. Shepp and Shiryaev (1993) proved the following result:

Let

$$\gamma_{1,2}=-\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)\mp\sqrt{\frac{2r}{\sigma^2}+\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2}\quad\text{and}\quad\alpha=\left(\frac{1-1/\gamma_1}{1-1/\gamma_2}\right)^{1/(\gamma_2-\gamma_1)}.$$

Then

$$\sup_T E\left(e^{-rT}M_T\right)=E\left(e^{-rT^*}M_{T^*}\right),$$

where $T^*=\inf\{t>0|M_t/S_t=\alpha\}$.

To obtain this result we first follow the approach of Shepp and Shiryaev (1994) and rewrite the problem appropriately. Let $X_t=M_t/S_t$. Let $\tilde{P}$ denote the probability measure given by

$$\frac{d\tilde{P}}{dP}\Big|_{\sigma(B_s, 0\le s\le t)} = e^{\sigma B_t - \frac{\sigma^2}{2}B_t} = e^{-\mu t}S_t.$$

Let $\tilde{E}$ denote the expectation with respect to $\tilde{P}$. Then for all stopping times T of S

$$E\Big(e^{-rT}M_T 1_{\{T<\infty\}}\Big) = E\Big(e^{-(r-\mu)T}e^{-\mu T}S_T X_T 1_{\{T<\infty\}}\Big) = \tilde{E}\Big(e^{-(r-\mu)T}X_T 1_{\{T<\infty\}}\Big).$$

The process X is a regular diffusion under $\tilde{P}$ with state space $[1,\infty)$ and instantaneous reflection at 1. The infinitesimal generator of X in $(1,\infty)$ is equal to

$$\frac{\sigma^2}{2}x^2\frac{\mathrm{d}}{\mathrm{d}x^2} - \mu x\frac{\mathrm{d}}{\mathrm{d}x}.$$

This means that X behaves like an exponential Brownian motion with drift $-\mu$, diffusion coefficient σ and instantaneous reflection at 1. To maximize $\tilde{E}(e^{-(r-\mu)T}X_T 1_{\{T<\infty\}})$ over all stopping times T of X, let

$$H(x) = \frac{\eta_2}{\eta_2-\eta_1}x^{\eta_1} - \frac{\eta_1}{\eta_2-\eta_1}x^{\eta_2}.$$

Here $\eta_1 < 0$ and $\eta_2 > 1$ are the two roots of

$$\frac{\sigma^2}{2}x^2 - \left(\mu + \frac{\sigma^2}{2}\right)x = (r-\mu).$$

Note that $\eta_{1,2} = 1 - \gamma_{2,1}$. Then

$$\frac{\sigma^2}{2}x^2 H''(x) - \left(\mu + \frac{\sigma^2}{2}\right)H'(x) = (r-\mu)H(x)$$

and $H'(1) = 0$. Moreover $H(x) \ge 0$ for $x \ge 1$ and $H(1) = 1$. So, $\exp\{-(r-\mu)t\}H(X_t)$ is a positive locale martingale. This yields for $a \ge 1$ that

$$\tilde{E}\left(e^{-(r-\mu)T_a}1_{\{T_a<\infty\}}\right) = \tilde{E}\left(e^{-(r-\mu)T_a}H(X_{T_a})\frac{1}{H(X_{T_a})}1_{\{T_a<\infty\}}\right) = \frac{1}{H(a)}.$$

Note that $\tilde{P}(T_a < \infty) = 1$ for all $a > 1$ (see Shepp and Shiryaev (1994)) and $0 \leq H(X_t) \leq \sup_{1 \leq x \leq a} H(x) < \infty$ for $0 \leq t \leq T_a$. The function $\frac{x}{H(x)}$ assumes its maximum over $[1, \infty)$ uniquely at $x^* = \alpha$. Therefore $T^* = T_{x^*}$. Theorem 2 i) now yields the assertion with $\psi_+(x) = H(x)$.

5 Proofs

Proof of Theorem 1. Suppose $\sup_{x \geq x_0, x \in I}[h(x)/\psi_+(x)] = +\infty$. Then there exists a sequence $x_n \geq x_0$ with $\lim_{n\to\infty}[h(x_n)/\psi_+(x_n)] = +\infty$. Let $S_n = T_{x_n}$. Lemma 1 and (5) yield

$$E_{x_0}\left(e^{-A(S_n)}h(X_{S_n})1_{\{S_n<\infty\}}\right) = \frac{h(x_n)}{\psi_+(x_n)} \ .$$

Hence

$$\sup_T E_{x_0}\left(e^{-A_T}h(X_T)1_{\{T<\infty\}}\right) \geq \sup_n \frac{h(x_n)}{\psi_+(x_n)} = +\infty \ .$$

A similar argument applies to the case $\sup_{x \leq x_0, x \in I}[h(x)/\psi_-(x)] = +\infty$.

Proof of Theorem 2 i). There exists a sequence $x_n \geq x_0$ with

$$\lim_{n\to\infty}[h(x_n)/\psi_+(x_n)] = \sup_{x\in I}\frac{h(x)}{\psi_+(x)} \ .$$

Let $S_n = T_{x_n}$. Lemma 1 and (5) yield

$$E_{x_0}\left(e^{-A(S_n)}h(X_{S_n})1_{\{S_n<\infty\}}\right) = \frac{h(x_n)}{\psi_+(x_n)} \ .$$

Hence

$$\sup_T E_{x_0}\left(e^{-A_T}h(X_T)1_{\{T<\infty\}}\right) \geq \sup_n \frac{h(x_n)}{\psi_+(x_n)} = \sup_{x\in I}\frac{h(x)}{\psi_+(x)} \ .$$

On the other hand (6) yields for any stopping time T of X that

$$E_{x_0}\left(e^{-A_T}h(X_T)1_{\{T<\infty\}}\right) = E_{x_0}\left(M_T^{(+)}\frac{h(X_T)}{\psi_+(X_T)}1_{\{T<\infty\}}\right) \leq \sup_{x\in I}\frac{h(x)}{\psi_+(x)} \ .$$

Now suppose $C^* = h(x^*)/\psi_+(x^*)$ for some $x^* \geq x_0$. Lemma 1 yields for the stopping time $T^* = \inf\{t > 0|\ X_t = x^*\}$ that

$$E_{x_0}\left(e^{-rT^*}h(X_{T^*})1_{\{T^*<\infty\}}\right) = C^*\left(E_{x_0}M_{T^*}^{(+)}\right) = C^*\ .$$

The proof of Theorem 2 ii) is similar to the proof of Theorem 2 i).

Proof of Theorem 3. There exist sequences $x_n^{(+)} \geq x_0$ and $x_n^{(-)} \leq x_0$ such that

$$\lim_{n\to\infty}\frac{h(x_n^{(+)})}{p^*\psi_+(x_n^{(+)}) + (1-p^*)\psi_-(x_n^{(+)})} = C^* = \lim_{n\to\infty}\frac{h(x_n^{(-)})}{p^*\psi_+(x_n^{(-)}) + (1-p^*)\psi_-(x_n^{(-)})}\ .$$

Let S_n denote the stopping time $S_n = \inf\{t \geq 0|X_t = x_n^{(+)} \quad \text{or} \quad X_t = x_n^{(-)}\}$. Obviously we have $S_n \leq \inf\{t \geq 0|X_t = x_n^{(+)}\}$ and $S_n \leq \inf\{t \geq 0|X_t = x_n^{(-)}\}$ and so Lemma 1 yields $E_{x_0}[(p^*M_{S_n}^{(+)} + (1-p^*)M_{S_n}^{(-)})1_{\{S_n<\infty\}}] = 1$. Hence

$$\begin{aligned} &E_{x_0}\left(e^{-A(S_n)}h(X_{S_n})1_{\{S_n<\infty\}}\right)\\ &\quad\geq \min\left\{\frac{h(x_n^{(+)})}{p^*\psi_+(x_n^{(+)}) + (1-p^*)\psi_-(x_n^{(+)})}, \frac{h(x_n^{(-)})}{p^*\psi_+(x_n^{(-)}) + (1-p^*)\psi_-(x_n^{(-)})}\right\}\end{aligned}$$

and consequently $\sup_T E_{x_0}\left(e^{-A(T)}h(X_T)1_{\{T<\infty\}}\right) \geq C^*$. On the other hand (6) yields

$$\sup_T E_{x_0}\left(e^{-A(T)}h(X_T)1_{\{T<\infty\}}\right) \leq \sup_{x\in I}\frac{h(x)}{p^*\psi_+(x) + (1-p^*)\psi_-(x)} = C^*\ .$$

6 Extensions

6.1 Absorbing boundaries

We can also treat the case of absorbing boundaries. Let us suppose that a is a left absorbing boundary point and that we have no right absorbing boundary point.

The cases

- of a right absorbing boundary point but no left absorbing boundary point
- of a right absorbing boundary point and a left absorbing boundary point

can be treated in a similar fashion.

Now $\psi_+(a) = 0$ and $\psi_-(x) > 0$ for all $x \in I$. The case $x_0 = a$ is trivial. Let $x_0 > a$. The situation where Theorem 2 ii) applies requires no changes. If $h(a) = 0$, one simply has to replace $\sup_{x \le x_0, x \in I}[h(x)/\psi_+(x)]$ and $\sup_{x \in I}[h(x)/\psi_+(x)]$ by $\sup_{a < x \le x_0, x \in I}[h(x)/\psi_+(x)]$ and $\sup_{x \in I, x > a}[h(x)/\psi_+(x)]$ respectively. Then one can apply Theorem 2 i) and Theorem 3. Therefore we assume

$$\sup_{x \ge x_0, x \in I} \frac{h(x)}{\psi_-(x)} > \sup_{x \le x_0, x \in I} \frac{h(x)}{\psi_-(x)} \tag{11}$$

and $h(a) > 0$.

We now have to look at two different cases seperately. Let $\tilde{h}(x) = h(x)1_{\{x > a\}}$. It holds for any stopping time T that

$$E_{x_0}\left(e^{-A_T} h(X_T) 1_{\{T < \infty\}}\right) \ge E_{x_0}\left(e^{-A_T} \tilde{h}(X_T) 1_{\{T < \infty\}}\right) \ .$$

Theorem 1 now yields $\sup_T E_{x_0}(e^{-A_T} h(X_T) 1_{\{T<\infty\}}) = +\infty$ if

$$\sup_{x \ge x_0, x \in I} [\tilde{h}(x)/\psi_+(x)] = \sup_{x \ge x_0, x \in I} [h(x)/\psi_+(x)] = +\infty \ .$$

If $\sup_{x \ge x_0, x \in I}[h(x)/\psi_+(x)] < \infty$, we obtain together with (11) the existence of some $p^* \in (0, 1)$ satisfying (7). That means we are now in the situation covered by Theorem 3. It is intuitively clear that Theorem 2 i) cannot apply if $h(a) > 0$ since for any $x > x_0$ we have

$$E_{x_0}\left(e^{A(T_x)} h(X_{T_x}) 1_{\{T_x < \infty\}}\right) < E_{x_0}\left(e^{A(T_x \wedge T_a)} h(X_{T_x \wedge T_a}) 1_{\{T_x \wedge T_a < \infty\}}\right) \ .$$

6.2 Terminal Times

We can also cover situations where the process X is eventually killed. Let ξ be a terminal time of X; that is a stopping time of X with the property that $\xi = s + \xi \circ \theta_s$ on the event $\{\xi > s\}$. After possibly switching to a smaller state space, we may assume that $P(T_y < \xi | X_0 = x) > 0$ holds for all $y \in I$ and $x \in \text{int}(I)$. Our method can be adapted to deal with the problem of finding a stopping time T that maximizes

$$E\left(e^{-A(T)} h(X_T) 1_{\{T<\xi\}}\right) .$$

In this case the functions ψ_+ and ψ_- in (3) and (4) have to be replaced by

$$\psi_+(x) = \begin{cases} E_x\left(e^{-A(T_{x_0})} 1_{\{T_{x_0}<\xi\}}\right) & \text{for} \quad x \leq x_0 \\ 1/E_{x_0}\left(e^{-A(T_x)} 1_{\{T_x<\xi\}}\right) & \text{for} \quad x \geq x_0 \end{cases} \tag{12}$$

and

$$\psi_-(x) = \begin{cases} 1/E_{x_0}\left(e^{-A(T_x)} 1_{\{T_x<\xi\}}\right) & \text{for} \quad x \leq x_0 \\ E_x\left(e^{-A(T_{x_0})} 1_{\{T_{x_0}<\xi\}}\right) & \text{for} \quad x \geq x_0 \end{cases} \tag{13}$$

6.3 Discounting with Functionals which assume negative values

One might also think of applying our approach to problems where A assumes positive and negative values. For a particular example where A_t changes sign see Section 2.6 of Beibel and Lerche (1997). In this situation we can still follow the arguments in Section 2 and construct the functions ψ_+ and ψ_-. Under suitable integrability conditions, like

$$E_x\left(e^{-A(T_y)} 1_{\{T_y<\infty\}}\right) < \infty$$

for all x, y the processes M_t^- and M_t^+ will become local martingales and Lemma 1 will hold. Then we are able to give analogues of Theorem 1 and Theorem 2 i) and ii). Unfortunately Lemma 2 will in general fail to hold and so we are not able to prove the existence of a p^* satisfying (7). Therefore we can not give an analogue of Theorem 3 in this case.

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