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An empirical study concerning the detection of trend changes in some financial time series

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Nr. 36 Jan 1997

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Abstract
Several statistics which are used for change-point detection are studied, when applied to DAX-data, to find changes of trends. Using the usual levels of significance, the time lag between a change of trend and its detection is quite long. This is in agreement with the theory of sequential statistics.

1. Introduction

The detection of changes of trends is a major task in analyzing financial time series. Usually technical analysists apply tools of data analysis which arose by intuition but have no theoretical statistical foundation.

On the other hand there is some statistical literature on sequential change-point detection, which can be applied to financial time series. Here we present a little study in this spirit of a rather friendly DAX-dataset. We call it friendly since it contains two time periods, where the market seem to have changed trends. In fall 1992 it had its low at 1420 points and then slowly turned around. In July 1993 there was another certain uprise. Both cases are considered here, where of course the latter one is the much harder to analyze. Beside others the Cusum-statistic is discussed. This is one of the few statistics which arose from statistical theory and is known to many technical analysists.

We start with explaining our statistical model in Section 2. Then we discuss in Section 3 how we determine our change-point a posteriori and how we standardize the data-set. In section 4 we present the various statistics together with some figures.

2. The model

We suppose that our data after standardizations fulfills the following model assumptions:
$X_1, X_2, \ldots, X_\tau, X_{\tau+1}, \ldots$ are independent normally distributed random variables with unit variance and mean zero up to time τ. After time τ, which may be either a fixed time point or a realization of a random variable, the X_i have drift θ, where θ denotes either a known real value or an outcome of a random variable. We denote the corresponding probability measures by $P_{\theta,k}$ and $P_{\theta,\tau}$, where k stands for a fixed time point and τ stands for a random variable interpreted as a prior distribution of the change point. We note that $P_{0,0}$ is the measure of no change.
We are aware of the fact that our model is far too simple to describe the real data precisely (see [2]). Nevertheless there is strong evidence by central limit arguments, that our conclusions are rather stable, when refining the underlying model.

3. Preparing the data

For our investigation we use daily sampled data of the German stock market index DAX, recorded between August 1992 and October 1993.
For X_i we take the increments of the prices and standardize them in an appropriate way. For standardization we choose a certain change-point τ_e and remove the drift of the observations before τ_e. By rescaling the variance of the observations becomes one. It should be remarked, that for doing this, we adopt a posterior point of view which means, that we use the whole dataset to find the most likely change-point τ_e. We use the following statistics:
Let $\bar{X}_1(k) = \frac{1}{k}\sum_1^k X_i$ and $\bar{X}_2(k) = \frac{1}{n-k}\sum_{k+1}^n X_i$. We shall consider

$$quad.res(k) := \sum_1^k \left(X_i - \bar{X}_1(k)\right)^2 + \sum_{k+1}^n \left(X_i - \bar{X}_2(k)\right)^2, \qquad k = 1, \ldots, n,$$

and

$$abs.res(k) := \sum_1^k |X_i - \bar{X}_1(k)| + \sum_{k+1}^n |X_i - \bar{X}_2(k)|, \qquad k = 1, \ldots, n.$$

A reasonable choice of the change-point τ_e from the posterior point of view may be a $\hat{\tau}_e$ which has low values for these statistics.
Another possibility is to perform multiple t-tests (see esp. [8] and [5]), and guess the change-point as that $\hat{\tau}_e$, where the following statistic attains its maximum:

$$t(k) := \sqrt{\frac{k(n-k)}{n}}\,\frac{|\bar{X}_1(k) - \bar{X}_2(k)|}{s_e(k)},$$

with

$$s_e^2(k) := \frac{1}{(n-2)} \left(\sum_{1}^{k} (X_i - \bar{X}_1(k))^2 + \sum_{k+1}^{n} (X_i - \bar{X}_2(k))^2 \right), \qquad k = 1, \ldots, n.$$

Under the hypothesis that no change occurred the asymptotic distribution of the maximum of $t(k)$ is available (see [8]) and therefore also the asymptotic p-values.

Figure 1 shows the raw data of the German DAX-index.

Fig. 1

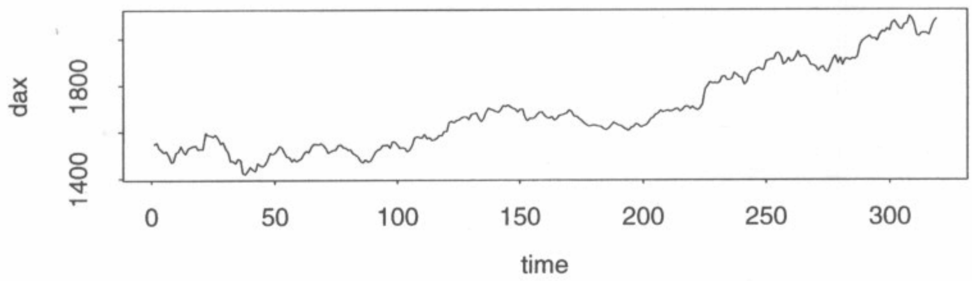
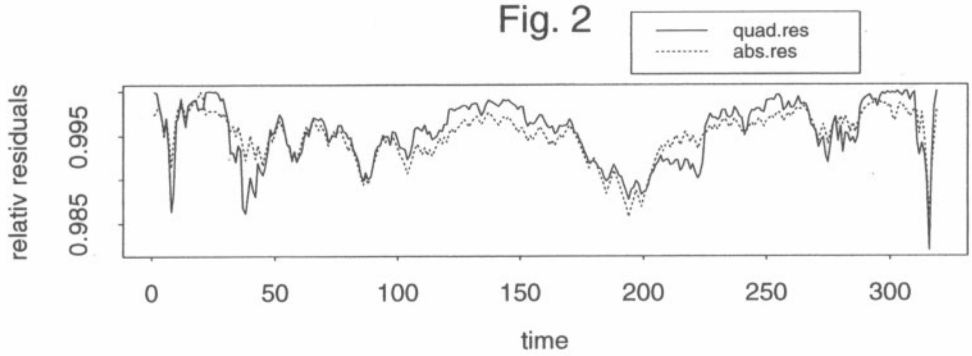


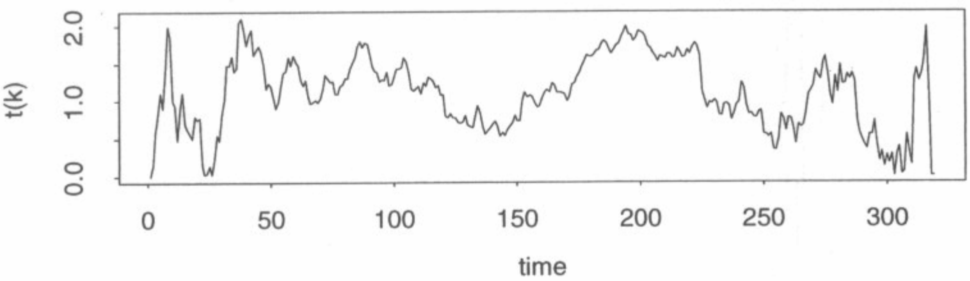
Figure 2 includes the values of *quad.res* and *abs.res* on an appropriate scale[1] and Figure 3 does this for the statistic $t(k)$.

Fig. 2



[1] We have decided to divide the values by their maximum over the whole range.

Fig. 3



From these statistics we choose our experimental change-point τ_e, which serves for standardizing the original data.

Fig.2 and Fig.3 resemble each other, which is not very surprising, since the statistics all deal with residuals in some sense. Apart from the two pathological peaks or holes at the very beginning and the end of the data set[2], the three figures suggest the most likely candidates for $\hat{\tau_e}$: $k = 38, k = 86$ and $k = 194$. All three statistics have $k = 86$ with a bit less evidence than the two others. The statistic $abs.res$ prefers $k = 194$ to $k = 38$, whereas the two others are in favour of $k = 38$[3]. It is interesting to note, that the t-value 2.09 which is reached at $k = 38$ has a p-value of 0.018, whereas the t-value 1.98 reached at $k = 194$ has p-value 0.024. Nevertheless it is misleading to take these p-values serious, since we have enjoyed the luxury of evaluating 317 t-tests. If we take an approximative expression of the correct p-value as in [8], we get about 0.5 for $k = 38$, so that the posterior-analysis cannot reject the hypothesis that there is no change at all.

But for the sake of watching the stopping times at work, we choose $\tau_e = 194$. Equipped with this standardizing-point we can perform as follows. With the rescaled increments,

$$Z_i = \frac{X_i - \bar{X}_1(\tau_e)}{s_e(\tau_e)}, \qquad i = 1, \ldots, n,$$

the standardized random-walk $S(k)$ is given by

$$S(k) := \sum_{i=1}^{k} Z_i, \qquad k = 1, \ldots, n.$$

For the DAX data we have a 'posterior drift' after τ_e of

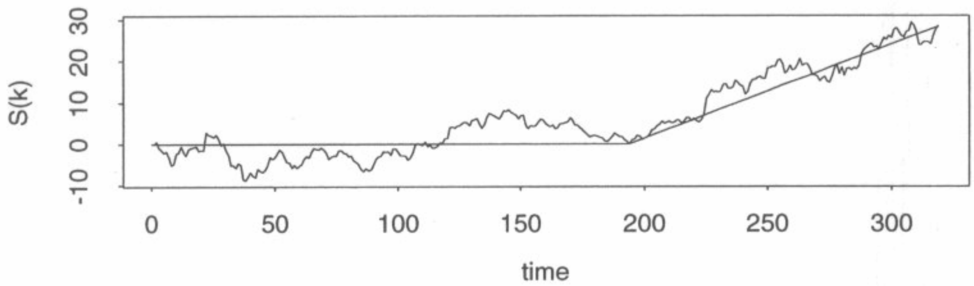
$$\theta_e = \frac{\bar{X}_2(\tau_e) - \bar{X}_1(\tau_e)}{s_e(\tau_e)} = 0.23 \tag{1}$$

[2]This comes from the asymptotic behavior of $t(k)$ near $k = 1$ and $k = n$ (see [8]).

[3]At $k = 38$ the DAX index attains its minimum over the whole underlying dataset.

units per day.

Fig. 4



Looking at Figure 4 it seems quite reasonable to assume a change-point at $\tau_e = 194$.

4. Sequential analysis

We consider now the standardized data as outcomes of a sequence of independent, normal random variables $X_1, \ldots, X_\tau, X_{\tau+1}, \ldots$, where $X_i, i \leq \tau$ are $N(0,1)$-distributed and $X_i, i > \tau$ are $N(\theta, 1)$-distributed[4].
We take the sequential point of view (which means that at time k only $X_1, \ldots, X_k$ are known) and analyze our data set with some sequential methods. They are based on stopping times of the time series $X_1, X_2, \ldots$. Here stopping means an alarm stating that the drift has changed. The stopping times under consideration all stop when certain statistics exceed some given thresholds. This means that they have the following structure:

$$T_{statistic}(b) := \inf\{t \geq 0 | statistic(t) \geq b\},$$

where statistic denotes one of the quantities defined below. Therefore the specific underlying statistics can be regarded as indicators for the occurrence of a change-point.

[4]In our case we choose of course $\tau = \tau_e$.

4.a. Statistics for simple hypothesis

Most of the stopping statistics proposed in the sequential literature are based on likelihood-ratio quantities. The following four procedures all treat the post-change drift θ as known, which is of course not realistic. Here we will take $\theta = \theta_e$.

The first statistic is known as Cusum-procedure and is motivated by maximum-likelihood consideration. It is given by

$$CUS^{\theta}(k) := \theta \left\{ S(k) - \frac{\theta}{2}k - \min_{j \leq k}(S(j) - \frac{\theta}{2}j) \right\}, \tag{2}$$

where $S(k)$ denotes the underlying random walk. $CUS(k)$ figures as maximum of the log-likelihood-ratios $\frac{dP_{\theta,\tau_e}}{dP_{0,0}}|_{\mathcal{F}_k}$ with $\mathcal{F}_k = \sigma(X_i | i \leq k)$, where k runs through all possible values from $1, \ldots, n$.

Another possibility is to regard the likelihood ratio $LIK^{\theta}(k) := \frac{dP_{\theta,\tau}}{dP_{0,0}}|_{\mathcal{F}_k}$, where we choose the prior distribution of τ arbitrarily. In what follows we use a geometric distribution $P_{\theta,\tau}(\tau = k) := q^k p$ with $q = (1-p)$. Thus $LIK^{\theta}(k)$ is given by [5]:

$$LIK^{\theta}(k) := \sum_{j=0}^{k} \exp\{\theta(S(k) - S(j)) - \frac{\theta^2}{2}(k-j)\} pq^j + P_{\theta,\tau}(\tau > k). \tag{3}$$

Another statistic closely related to LIK^{θ} is the posterior probability $POST(k) = \pi^{\theta}(k) = P_{\theta,\tau}(\tau \leq k | S(j), j \leq k)$, which is computed as

$$\pi^{\theta}(k) = \frac{LIK^{\theta}(k) - P_{\theta,\tau}(\tau > k)}{LIK^{\theta}(k)}.$$

Therefore, by drastically exceeding the prior probability $PRIOR(k) = P_{\theta,\tau}(\tau \leq k)$, the posterior $POST(k)$ gives some evidence for the occurrence of a change point.

By letting the parameter p of the prior distribution tend to zero and so allowing the prior distribution to become improper, we derive from the statistic considered above the well known Shiryayev-Roberts-statistic (see e.g. [9]), which is

$$SR^{\theta}(k) := \sum_{j=0}^{k} \exp\{\theta(S(k) - S(j)) - \frac{\theta^2}{2}(k-j)\}.$$

[5]This approach relies on a Bayesian point of view, which seems advantageous in this context, since it allows to get rather simple expressions of a risk structure and therefore solvable optimization problems, see [13] and [1].

Since under the hypothesis $P_{0,0}$ the statistic $SR^{\theta}(k) - k$ is a martingale we expect that $SR^{\theta}(k)$ takes values much higher than k, when a change already occurred.
In the following we present some graphs of the above statistics for the rescaled DAX-data with $\theta = \theta_e$. For *LIK* and *POST* we choose p to fulfill $E_{\theta,\tau}\tau = \tau_e$.

Fig. 5

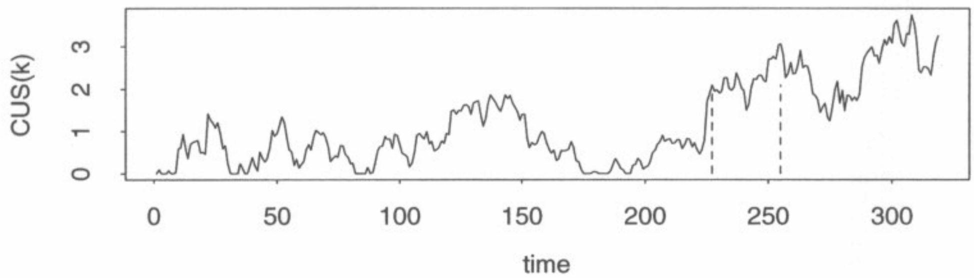


Fig. 6

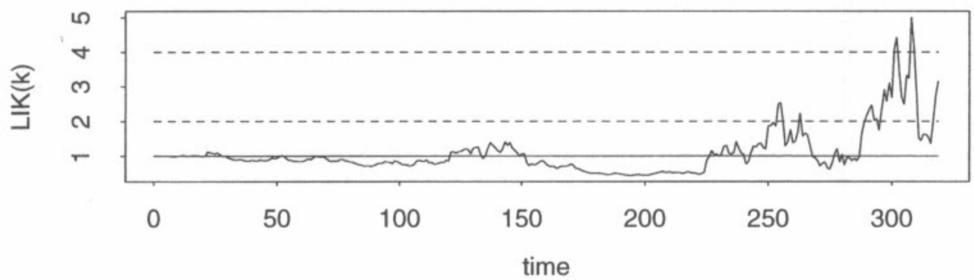
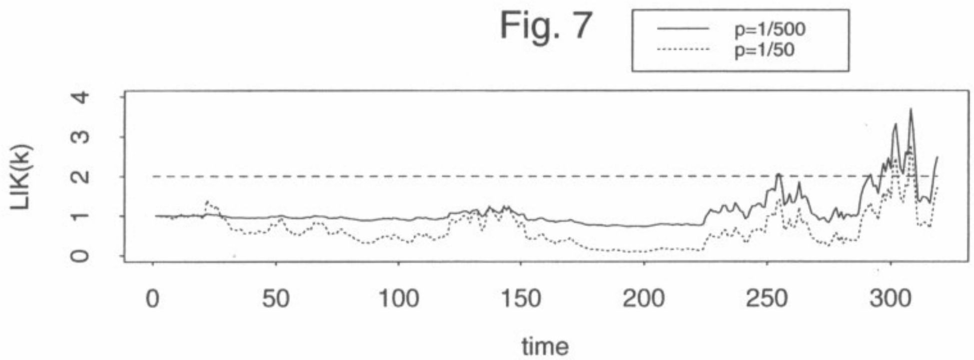


Fig. 7



To see a little bit more clearly what happens it seems reasonable to transform $POST(k)$ and $SR(k)$ on a more convenient scale. Therefore we consider the difference between $POST(k)$ and $PRIOR(k)$ (which is indicated in Fig. 8 with the dashed line) relatively to $(1 - PRIOR(k))$ instead of the mere

$POST(k)$. So we get

$$rel.POST(k) := \frac{POST(k) - PRIOR(k)}{1 - PRIOR(k)}.$$

For $SR(k)$ we consider

$$rel.SR(k) = \frac{SR(k) - k}{k}.$$

Fig. 8

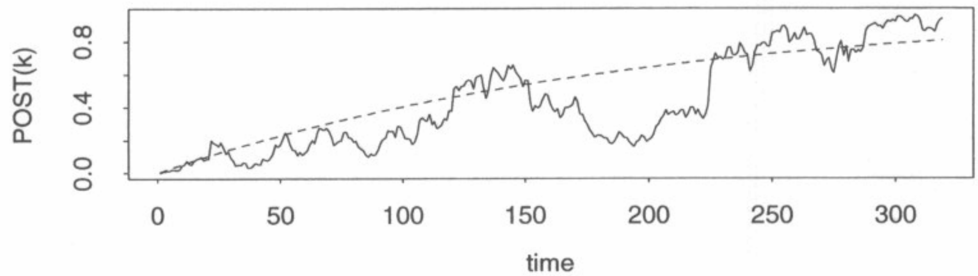


Fig. 9

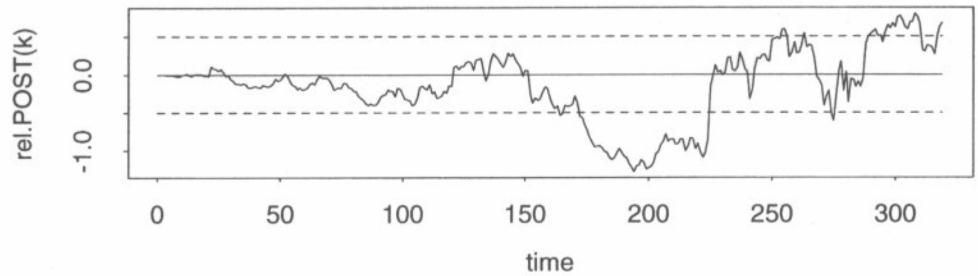


Fig. 10

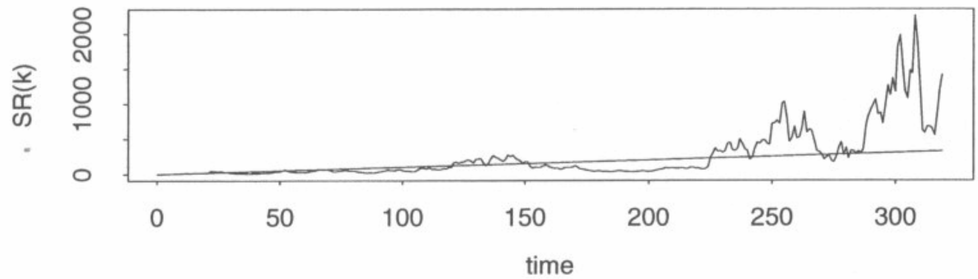
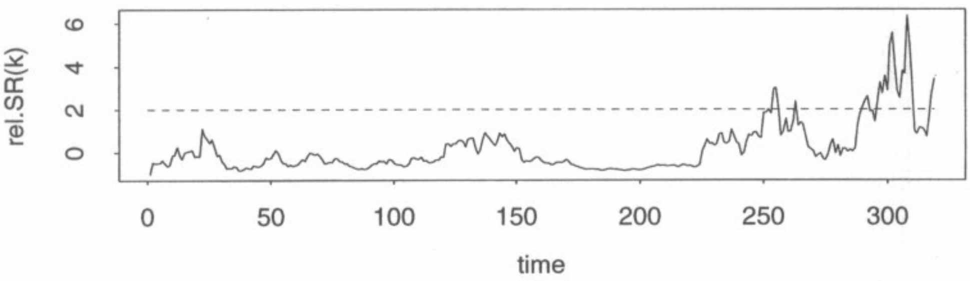


Fig. 11



What can we conclude from these plots? Fig. 5 looks slightly more impressive then Fig. 4 itself. The two most remarkable jumps upwards at $k = 227$ and $k = 255$ are better visible than in the original data (Fig. 4).
In Fig. 7 we vary the parameter p which was in Fig. 6 chosen to hold the equation $E_{\theta,\tau}\tau = \tau_e$. Since the LIK-statistic stems from the likelihood-ratio of two distinct probability-measures, the values of LIK are closely related to error probabilities. Thus to use the LIK-statistic means to work confirmatively in the common statistical sense. According to the classical theory, stopping at $b = 2$ ($b = 4$) means to reject the hypothesis of no change with an error probability of about 50% (25% resp.). In face of the levels usually used in confirmative statistics the significance level achieved with $LIK(255) = 2.54$ looks in no way overwhelming. Fig. 7 with the other prior distributions shows that for the early prior with $E_{\theta,\tau}\tau = 50$, the time point $k = 255$ loses any importance since it does not contrive in reaching the level $b = 2$. The progression of the LIK-statistic using the late prior corresponding to $E_{\theta,\tau}\tau = 500$ resembles Fig. 6, the plotted line moves on a slightly lower level.
Fig. 8 or the rescaled Fig. 9 also suggest to stop at $k = 255$ and so does the Shiryayev-Roberts statistic. In Fig. 11 $k = 255$ is the first time, where the $rel.SR$-statistic crosses level 2.
Briefly one can resume as following: all the above statistics suggest $k = 255$ in a more or less expressive way, but the phrase of high significance should not be used in this context. On the other hand the delay of 61 trading days, which means nearly three months delay, looks rather daunting. But a change of 0.23 units per day is also quite hard to detect. For example, an approximation for the delay of the stopping times $T_{LIK^\theta}(b)$ is given by (see [4])

$$E_{\theta,\tau}(T_{lik^\theta}(b) - \tau)^+ \approx \frac{2}{\theta^2}(\log b + K(p,\theta)), \tag{4}$$

with K a constant depending of the prior parameter p and θ. We evaluated K approximatively following the lines of [4]. This yields an expected delay

of 85 days with $b = 2.5$.
The study for the change-point chosen at $k = 38$ looks somewhat different. In this case we get $\theta_e = 0.37$, which yields a simpler detection problem. The following pictures support this impression:

Fig. 4b

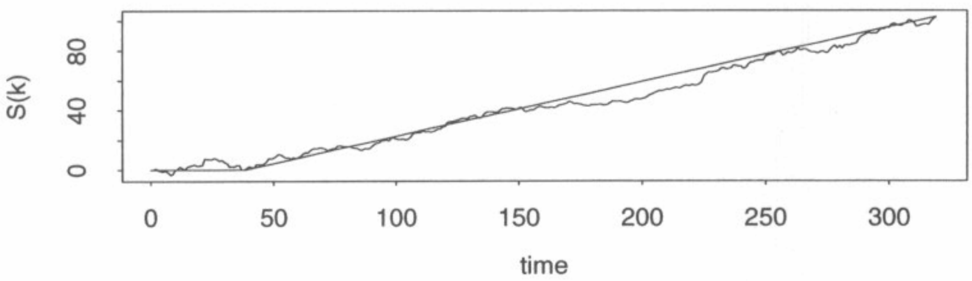


Fig. 5b

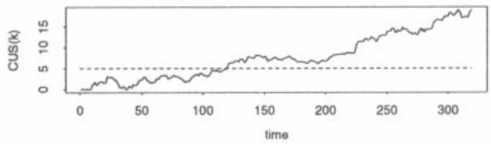


Fig. 6b

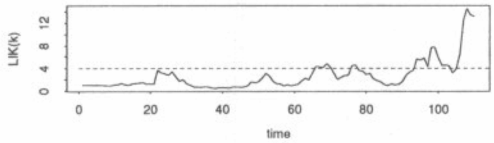


Fig. 8b

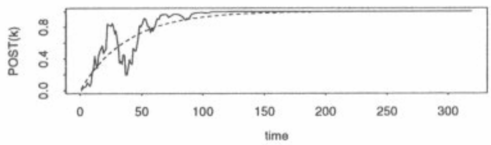


Fig. 10b

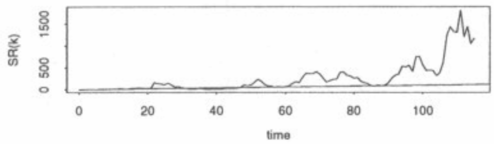
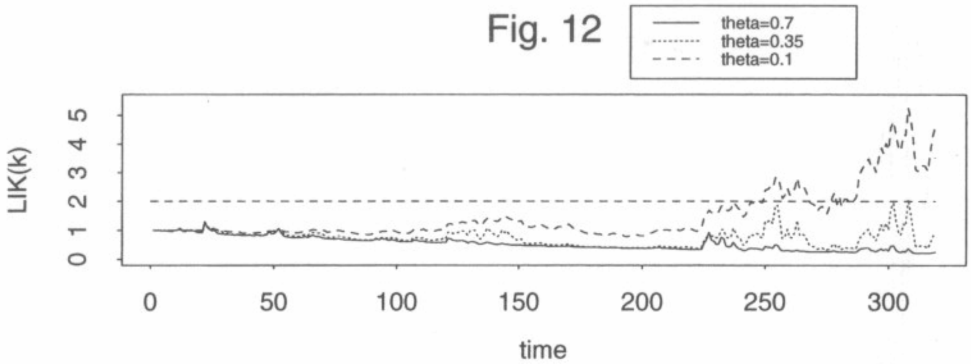


Fig. 4b shows that the progression of the standardized data with $\tau_e = 38$ follows the regression line quite regularly. The Cusum-statistic reflects this by showing nearly the same shape as the original curve, whereas the LIK^θ- and SR^θ statistics grow very rapidly after $k = 107$, e.g. LIK passes $b = 10$ at $k = 107$, $b = 20$ at $k = 119$ and $b = 100$ at $k = 126$. The approximation formula (4) renders for $b = 4$ an expected delay of about 33 days and LIK passes $b = 4$ at $k = 66$. In this case the common values for safeness in statistical inference are reached around $k = 120$, which means a delay of about 80 trading days.

4.b. Statistics for composite hypothesis

All the above statistics deal with the fixed parameter θ_e and the stopping times based on Lik and SR are known to work at their best since θ_e is equal to the posterior drift (see [13], [10]). Thus the statistics should also work quite good, when θ is near the right value of the post-change drift. In Fig. 12 we vary the 'known' drift θ_e fed into the LIK^θ-statistic applied to our random walk $S(k)$, $k \geq 1$.

Fig. 12



The message seems to be clear. The higher the possible change (our guessed θ) is expected to be, the lower is the sensibility of the LIK statistic for our data set, since the indicator line moves generally downwards.

One possibility to get rid of the parameter θ_e, on which we rely in our exploration, is lent from the theory of tests with power one (see [7]). We regard the parameter θ as random, ruled by a distribution $G(d\theta)$ with $g(\{0\}) = 0$ and then simply mix the statistics according to that distribution. Since for every θ the LIK^{θ} statistic is a test of power one for the simple alternative of $P_{\theta,\tau}$, the 'mixed-statistic' inherits this property. For our practical evaluations we assume throughout the following, G to be normal $N(0,1)$.

Mixing the Cusum-statistic makes only sense, when we remove the log, that means to mix the expressions $exp(CUS^{\theta}(k))$. Thus one gets

$$mix.CUS(k) := \frac{1}{\sigma} e^{\frac{\mu^2}{2\sigma^2}} \max_{j \le k} \frac{1}{\sqrt{(k-j)+\sigma^{-2}}} \exp\left\{ \frac{(S(k)-S(j)+\frac{\mu}{\sigma^2})^2}{2(k-j+\sigma^{-2})} \right\}.$$

When we mix the likelihood-ratios we get

$$mix.lik(k) := \int_{\mathcal{R}} \sum_{j=0}^{k} \exp\{y(S(k)-S(j)) - \frac{y^2}{2}(k-j)\} pq^j G(dy) + P(\tau > k).$$

This stopping statistic has a known, nice optimal property, which is discussed in [1].

The same idea can be applied to the Shiryayev-Roberts-statistic. We get

$$mix.SR(k) := \int_{\mathcal{R}} \sum_{j=0}^{n} \exp\{y(S(k)-S(j)) - \frac{y^2}{2}(k-j)\} G(dy).$$

Under the measure $P_{0,0}$ this statistic inherits the martingale-property from its simple counterpart (see 4.a.). For the continuous case this is discussed in

some length in [10].
The following three pictures show these statistics when G has a $N(0,1)$-distribution.

Fig. 13

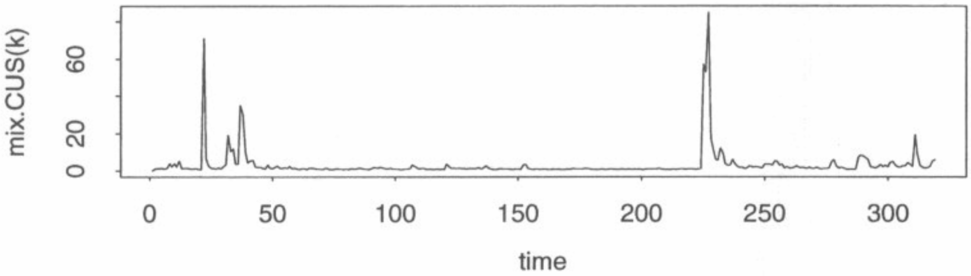


Fig. 14

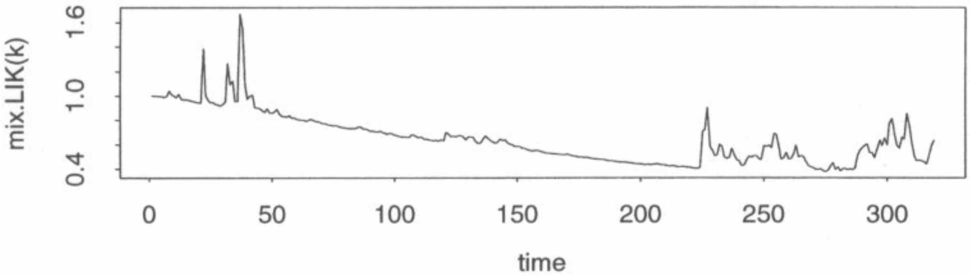
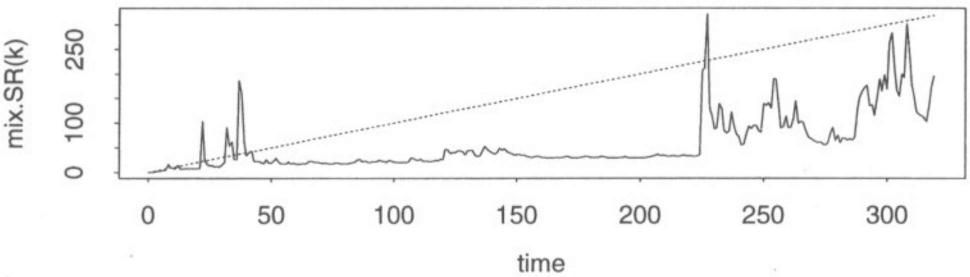


Fig. 15



It can be shown (see [1]) that the approximation formula (4) which is true for $T_{LIK^\theta}(b)$ is also valid for the mixed statistic $T_{mix.LIK}(b)$ up to terms of order $\log\log b$. (In [10] an analogous behavior is shown for $T_{SR^\theta}(b)$ and $T_{mix.SR}(b)$.) For our first data set with standardization point $\tau_e = 194$ Figs. 13-15 show, that the mixture-statistics have some difficulties in gaining some expressiveness. It seems that $mix.CUS$ and $mix.SR$ react very sensitive on a big jump upwards in the original data. This explains the peaks at the times $k = 22$ and $k = 227$. The $mix.lik$-statistic shows no reaction of that type until the end of our observation horizon at $n = 319$.

On the other hand for the second case with $\tau_e = 38$, when there is time enough, the mixture-statistics work similar to their simple counterparts.

Fig. 14a

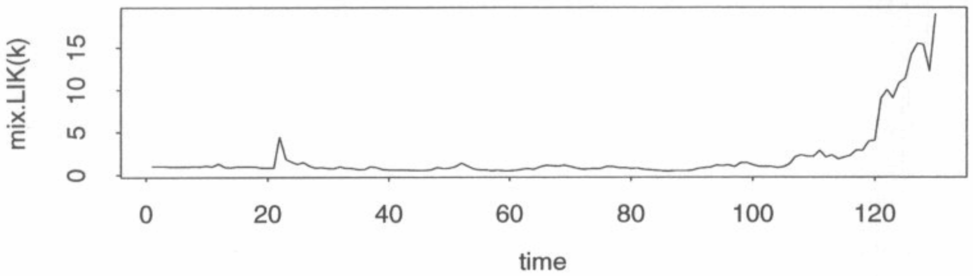


Fig. 13a

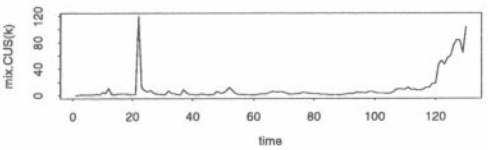
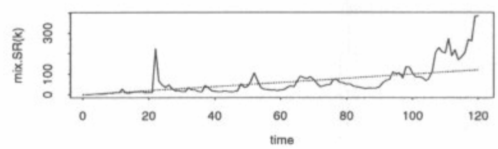


Fig. 15a



5. Conclusion

Since quickness and safeness are two contrary goals in change-point detecting, we cannot expect to get a unique procedure which covers all situations. There are problems, where quickness is advantageous, others where safeness is wished. The statistics investigated in this study all rely on likelihood considerations and focus more the side of safeness than that of quickness. But since the observed levels of safeness are not that high, one might be a bit unsatisfied. On the other hand, the statistics are in some sense the best and one must say that confirmative statements about drift-changes cannot be done better.

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