A Generalized Parking Problem

Michael Woodroofe*[1], Rudi Lerche[2] and Robert Keener[1]*

[1] University of Michigan
[2] Universitat Freiburg

Abstract. Let F denote a distribution function which has a finite positive mean μ; let $X_1, X_2, \ldots$ denote independent random variables with a common distribution function F; let $S_0, S_1, S_2, \ldots$ denote the random walk $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n = 1, 2, \ldots$; and let g denote a nonnegative finite convex function which attains its minimum at a unique $b \geq 0$. The problem of minimizing $E[g(S_t)]$ with respect to stopping times t is considered. It is shown that there is an $a < b$ for which it is optimal to stop as soon as $S_n > a$; and a is characterized in terms of a ladder height distribution.

1 Introduction

Let $X_1, X_2, \ldots$ denote independent and identically distributed random variables with a common distribution function F and a finite, positive mean μ, and let $S_n = X_1 + \cdots + X_n$, $n = 0, 1, 2, \ldots$ denote the random walk. Next, let g be a nonnegative, finite, convex function which attains its minimum at a unique $b \geq 0$, and regard $g(S_n)$, $n = 0, 1, 2, \ldots$ as a sequence of potential losses. The problem of minimizing $E[g(S_t)]$ with respect to stopping times t is considered.

Special cases of this problem have been solved elsewhere. The case in which $g(x) = |x - b|$ and F is a geometric distribution is called the parking problem by Chow, Robbins, and Siegmund (1971, p. 45, 60). Cases in which $g(x) = e^{-x} + cx$, where $0 < c < 1$ and $\int_{\mathbf{R}} e^{-x} F(dx) = 1$ arise in the construction of optimal sequential tests. Such cases were considered by Lorden (1977). Lorden's problem also arises in the work of Lerche (1991).

The purpose of this paper is to solve the problem under minimal conditions on g and F. In fact, it is not even assumed that $E[g(S_k)] < \infty$ for all (or any) k (when $F(0) > 0$). The solution is presented in Sect(s) 4 and 5, where it shown to be optimal to stop as soon as $S_n > a = a(F, g)$, and illustrated by examples in Sect. 6. When $F(0) = 0$, the problem is a simple monotone case. The reduction of the general to the special case uses the Wiener-Hopf factorization for random walks.

2 Preliminaries

To increase the generality of the results suppose that there are nondecreasing sigma algebras $\mathcal{A}_0, \mathcal{A}_1, \ldots$ for which X_k is $\mathcal{A}_k$-measurable and independent of $\mathcal{A}_{k-1}$ for all $k = 1, 2, \ldots$, and consider (a.e. finite) stopping times $t \geq 0$ with respect to $\mathcal{A}_0$, $\mathcal{A}_1, \ldots$. Of course, the sequence of sigma algebras generated by $X_1, X_2, \ldots$ satisfies these conditions.

* Research supported by the National Science Foundation.

A Truncation Lemma

From the standing assumptions on g it follows directly that

$$|x| = O(g(x)) \text{ as } |x| \to \infty. \tag{1}$$

In addition, the following related conditions are imposed in places:

$$g(x) = O(|x|) \text{ as } x \to -\infty, \tag{2}$$

$$g(x) = O(|x|) \text{ as } x \to \infty. \tag{3}$$

Lemma 1 *Suppose that (2) holds. If t is a stopping time and $E[g(S_t)] < \infty$, then*

$$\lim_{n\to\infty} \int_{t>n} g(S_n)dP = 0$$

and

$$E[g(S_t)] = \lim_{n\to\infty} E[g(S_{t\wedge n})],$$

where $t \wedge n$ denotes the minimum of t and n.

Proof. By (1), there is a constant C for which $E|S_t| \le C\{1 + E[g(S_t)]\} < \infty$. So, $E(t) < \infty$, by a converse to Wald's Lemma. See Gut (1988, Theorem I.5.5). Also, by (2), there is a constant M for which $g' \ge -M$, where g' denotes the right hand derivative. If $n \ge 1$ and $c > b$, then

$$\int_{t>n, S_n \le c} [g(S_t) - g(S_n)]dP \ge \int_{t>n, S_n \le c} g'(S_n)(S_t - S_n)dP, \tag{4}$$

by the convexity of g. On $\{S_n \le c\}$, $|g'(S_n)(S_t - S_n)| \le [M + g'(c)]|S_t - S_n|$, which is integrable; and $E[g'(S_n)(S_t - S_n)|\mathcal{A}_n] = g'(S_n)\mu E(t - n|\mathcal{A}_n)$, by Wald's Lemma. So, the right side of (4) becomes

$$\int_{t>n, S_n \le c} g'(S_n)\mu(t-n)dP \ge -M\mu \int_{t>n, S_n \le c} (t-n)dP.$$

Combining the last two expressions and letting $c \to \infty$, leads to

$$\int_{t>n} g(S_n)dP \le \int_{t>n} [g(S_t) + M\mu(t-n)]dP$$

for all $n \ge 1$. The first assertion of the lemma now follows by letting $n \to \infty$; and the second is an easy consequence of the first.

Some Notation

Convolution Operators. If G is a distribution function and h is a measurable function, write

$$Gh(x) = \int h(x+y)G(dy)$$

whenever the integral exists.

Ladder Variables. Let $\eta_0, \eta_1, \eta_2, \dots$ denote the (strict ascending) *ladder epochs,* defined by $\eta_0 = 0$ and

$$\eta_k = \inf\{n > \eta_{k-1} : S_n > S_{\eta_{k-1}}\},$$

for $k = 1, 2, \dots$. Write η for η_1. Then

$$E(\eta) = \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} P[S_n \le 0]\right\} < \infty$$

and

$$E(S_\eta) = \mu E(\eta) < \infty$$

by Sptizer's Identity and Wald's Lemma. See, for example, Woodroofe (1982, p. 22). Let H^+ denote the distribution function of the first *ladder height* S_η,

$$H^+(x) = P[S_\eta \le x]$$

for all $0 < x < \infty$. The *ladder heights* S_{η_k}, $k = 0, 1, 2, \dots$, form a random walk with step distribution H^+.

Assumption. It is assumed throughout that

$$E[g(S_\eta)] = H^+g(0) < \infty. \tag{5}$$

A convex function which satisfies (5) and the conditions listed in the introduction is called *admissible* below. If (5) fails then stopping immediately is optimal: see the remark following the proof of Theorem 2.

3 Lemmas

Some Properties of H^+g

The solution to the stopping problem requires some simple properties of H^+g.

Lemma 2

i) $H^+g(x) - g(x)$ *is nondecreasing in* x;
ii) H^+g *is continuous on the interior of* $\{x : H^+g(x) < \infty\}$;
iii) $H^+g(x) - g(x) > 0$ *for all* $x \ge b$;
iv) $\lim_{x\to-\infty} H^+g(x) - g(x) < 0$.

Proof. i) and iii) follow easily since

$$H^+g(x) - g(x) = \int_0^\infty [g(x+y) - g(x)]H^+(dy),$$

$g(x+y) - g(x)$ is nondecreasing in x for all $y > 0$, and $g(x+y) - g(x) > 0$ for all $x \geq b$ and $y > 0$. Next, ii) follows easily from the Dominated Convergence Theorem and the monotonicity of $g(x+y) - g(x)$. For iv) observe that for all $y > 0$, $g(x+y) - g(x)$ decreases as x decreases and that $\lim_{x \to -\infty} g(x+y) - g(x) < 0$, since $g(x+y) - g(x) < 0$ for all $x < b - y$. Then iv) follows directly from the Monotone Convergence Theorem.

For the next lemma, let

$$a = \sup\{y : H^+g(y) - g(y) < 0\}. \tag{6}$$

Lemma 3

i) $-\infty < a < b < \infty$;
ii) $H^+g(y) - g(y) \geq 0$ *for all* $y > a$;
iii) $H^+g(y) - g(y) < 0$ *for all* $y < a$;
iv) $H^+g(a) \leq g(a)$ *with equality if* $H^+g(x) < \infty$ *for some* $x > a$.

Proof. That $-\infty < a \leq b$, ii) and iii) are clear from (6) and Lemma 2. For the first part of iv), let $a_m = a - 1/m$ for $m = 1, 2, \ldots$. Then $H^+g(a_m) \leq g(a_m)$ for all $m \geq 1$, so that

$$0 \geq \liminf_{m \to \infty}[H^+g(a_m) - g(a_m)] \geq H^+g(a) - g(a)$$

by Fatou's Lemma, since $g(a_m + y) - g(a_m)$ is bounded below in $m \geq 1$ and $y \in \mathbb{R}$. The second part of iv) follows directly from Lemma 2-ii). Finally, that $a < b$ in i) follows from iv) and Lemma 2-iii).

Lemma 4 *If $\tilde{g}$ is another admissible convex function for which $\tilde{g}(y) = g(y)$ for all $y \geq a - 1$, then $\tilde{a} = a$, (where $\tilde{a}$ is defined by (6) with g replaced by $\tilde{g}$).*

Proof. This is immediate from Lemma 3 since the functions $H^+g - g$ and $H^+\tilde{g} - \tilde{g}$ are increasing, agree on $[a-1, \infty)$, and are negative on $[a-1, a)$ and nonnegative on (a, ∞).

First Passage Times

For all $c \in \mathbb{R}$, let

$$\tau_c = \inf\{n \geq 0 : S_n > c\}$$
$$\sigma_c = \inf\{n \geq 1 : S_n > c\}.$$

Observe that each τ_c is a ladder epoch and that $\sigma_c = \tau_c$ for all $c \geq 0$, and let

$$\rho(x) = E[g(x + S_{\tau_{a-x}})], \ x \in \mathbb{R}. \tag{7}$$

Lemma 5 *$\rho(x) < \infty$ for all $x \in \mathbb{R}$.*

The proof of this lemma is presented in Sect. 7.

It is shown below that τ_a is optimal. The need to consider σ_c as well as τ_c arises from the relation

$$F\rho(x) = E[g(x + S_{\sigma_{a-x}})], \tag{8}$$

for all $x \in \mathbb{R}$, which is easily verified by conditioning on X_1. This relation is exploited in Corollary 2 below.

4 The Monotone Case

The case in which $F(0) = 0$ is considered first. Then $H^+ = F$.

It is necessary to take conditional expectations of random variables whose expectations may be infinite (though defined). The definitions and conventions of Neveu (1965, Section IV.3) are used in such cases. With these conventions, it is clear that

$$E[g(x + S_{k+1}) - g(x + S_k)|\mathcal{A}_k] = Fg(x + S_k) - g(x + S_k) \tag{9}$$

for all $k \geq 0$ and all $x \in \mathbb{R}$. By Lemma 2 and the assumption that $F(0) = 0$, the right side of (9) is nondecreasing in k w.p.1. So, the stopping problem is like the *monotone case*, described by Chow, Robbins, and Siegmund (1971, Ch. 3) (who require more integrability than is assumed here). Observe that the right side of (9) is nonpositive for $k < \tau_{a-x}$ and nonnegative for $k \geq \tau_{a-x}$. Any stopping time τ with these properties is called a *monotone case rule (for x)* below.

Theorem 1 *Suppose that $F(0) = 0$. If $x \in \mathbb{R}$, then $E[g(x + S_t)]$ is minimized by $t = \tau_{a-x}$. In fact, $E[g(x + S_t)]$ is minimized by any monotone case τ.*

Proof. Following the outline of the proof of Theorem 3.3 of Chow, Robbins, and Siegmund (1971) it may be shown that for any for any monotone case stopping time τ

$$\int_A [g(x + S_t) - g(x + S_\tau)]dP = \int_A \left\{ \sum_{k=\tau\wedge t}^{\tau\vee t-1} |Fg(x + S_k) - g(x + S_k)| \right\} dP \tag{10}$$

for all $A \in \mathcal{A}_{\tau\vee t}$ for any stopping time t for which $E[g(S_t)] < \infty$, where $a\vee b$ denotes the larger of a and b. See Sect. 7 for the details. Theorem 1 is an immediate consequence.

Corollary 1 *If $F(0) = 0$, then $\rho(x) = \inf_t E[g(x + S_t)]$ for all $x \in \mathbb{R}$.*

5 The General Case

In the general case, where $F(0)$ may be positive, Theorem 1 may be applied to the ladder height process S_{η_k}, $k = 0, 1, 2, \ldots$. Since 0 and η are both ladder epochs, it follows that

$$\rho(x) \leq \min[g(x), H^+g(x)], \; \forall\, x \in \mathbb{R}. \tag{11}$$

Extension of Theorem 1 to the general case will be accomplished by considering the stopping times $\gamma_0 = 0$,

$$\gamma_k = \inf\{n > \gamma_{k-1} : S_n > S_{\gamma_{k-1}} + c\}, \; k = 1, 2, \ldots$$

and

$$\kappa = \inf\{k \geq 0 : S_{\gamma_k} > 0\},$$

where $c < 0$ is regarded as fixed for the moment. Observe that $\gamma_1 = \sigma_c$. It is easily seen that

$$\eta = \gamma_\kappa. \tag{12}$$

Proposition 1 *If $c < 0$ and $x \in \mathbb{R}$, then*

$$E[g(x + S_{\sigma_c})] \geq \min[g(x), H^+g(x)].$$

Proof. Fix $c < 0$ and $x \in \mathbb{R}$ throughout the proof. Let F_c denote the distribution function of $S_{\sigma_c} = S_{\gamma_1}$, and let H_c^+ and H_c^- denote the distribution functions of the strict ascending and weak descending ladder heights for a random walk with step distribution F_c. Here H_c^- is a defective distribution, and $H_c^+ = H^+$, by (12). So,

$$F_c = H^+ + H_c^- - H^+ * H_c^-,$$

where $*$ denotes convolution, by Wiener Hopf Factorization. See Feller (1971, pp. 570–571). Suppose first that (3) holds. Then $H^+g(x) < \infty$, since $E(S_\eta) < \infty$. So, since $E[g(x + S_{\sigma_c})] = F_c g(x)$ and H_c^- is supported by $[c, 0]$,

$$\begin{aligned} E[g(x + S_{\sigma_c})] &= H^+g(x) + H_c^- g(x) - H^+ * H_c^- g(x) \\ &= H^+g(x) + \int_0^\infty \int_c^0 [g(x+y) - g(x+y+z)] H_c^-(dy) H^+(dz). \end{aligned}$$

Here $g(x + y) - g(x + y + z)$ is nonincreasing in $y \leq 0$ for all $0 < z < \infty$, so that $g(x+y) - g(x+y+z) \geq g(x) - g(x+z)$ for all $y \leq 0$ and $0 < z < \infty$. Let $\alpha = H_c^+[c, 0]$. Then

$$\begin{aligned} E[g(x + S_{\sigma_c})] &\geq H^+g(x) + \int_0^\infty \int_c^0 [g(x) - g(x+z)] H_c^-(dy) H^+(dz) \\ &= H^+g(x) + \alpha[g(x) - H^+g(x)] \\ &\geq \min[g(x), H^+g(x)]. \end{aligned}$$

For the general case, let $h_m = -m$, g', or m accordingly as $g' < -m$, $-m \leq g' \leq m$, or $g' > m$, and let

$$g_m(y) = g(b) + \int_b^y h_m(x)\,dx \tag{13}$$

for all $y \in \mathbb{R}$ and $m = 1, 2, \ldots$ with the sign conventions governing Riemann integrals. Then each g_m is an admissible convex function which satisfies (3), and g_m increases to g as $m \to \infty$. So,

$$E[g(x + S_c)] \geq E[g_m(x + S_c)] \geq \min[g_m(x), H^+ g_m(x)]$$

for all $m = 1, 2, \ldots$. The proposition now follows by letting $m \to \infty$.

Corollary 2 *If $x \in \mathbb{R}$, then $F\rho(x) \geq \rho(x)$.*

Proof. If $x \leq a$, then $F\rho(x) = \rho(x)$, since $\sigma_{a-x} = \tau_{a-x}$. See (8). If $x > a$, then

$$F\rho(x) = E[g(x + S_{\sigma_{a-x}})] \geq \min[g(x), H^+ g(x)] \geq \rho(x),$$

by (11) and Proposition 1.

Theorem 2 *$E[g(S_t)]$ is minimized among all stopping times t by $t = \tau_a$.*

Proof. Since $E[g(S_{\tau_a})] = \rho(0)$, it suffices to show that $E[g(S_t)] \geq \rho(0)$ for all stopping times t; and the latter is clear if $E[g(S_t)] = \infty$.

Suppose first that (2) holds. By Corollary 2, $\rho(S_n), n = 0, 1, 2, \ldots$ is a submartingale. So,

$$E[g(S_t)] \geq E[\rho(S_t)] \geq \rho(0) = E[g(S_{\tau_a})]$$

for any bounded stopping time t by the Optional Stopping Theorem and (7). So, if t is any stopping time for which $E[g(S_t)] < \infty$, then

$$E[g(S_t)] = \lim_{n \to \infty} E[g(S_{t \wedge n})] \geq \rho(0),$$

by Lemma 1.

For the general case, in which (2) is not required, let $h_m = -m$ or g' accordingly as $g' \leq -m$ or $g' > -m$, and define g_m by (13) for $m = 1, 2, \ldots$. Then each g_m is an admissible convex function for which (2) holds, and g_m increases to g as $m \to \infty$. Define a_m and ρ_m by (6) and (7) with g replaced by g_m for all $m = 1, 2, \ldots$. If m is sufficiently large, then $g_m(y) = g(y)$ for all $y > a - 1$, so that $a_m = a$ and $\rho_m = \rho$. For any such m and any stopping time t,

$$E[g(S_t)] \geq E[g_m(S_t)] \geq \rho_m(0) = \rho(0),$$

to complete the proof.

Remark. If $H^+ g(0) = \infty$ then stopping immediately is optimal. This follows from Theorem 2 provided there is a convex function $\tilde{g} \leq g$ with $\tilde{g}(0) = g(0)$, $H^+ \tilde{g}(0) < \infty$ and $\tilde{a} < 0$, for then $\inf_t E[g(S_t)] \geq \inf_t E[\tilde{g}(S_t)] = \tilde{g}(0) = E[g(S_0)]$. If g_m is defined as in the proof of Proposition 1 then $H^+ g_m(0) \to \infty$ as $m \to \infty$, so for m_0 sufficiently large $\tilde{g} = g_{m_0}$ will satisfy $H^+ \tilde{g}(0) > \tilde{g}(0)$. Since $\tilde{g}$ grows linearly at $\pm\infty$, $H^+ \tilde{g}$ is everywhere finite. Hence $H^+ \tilde{g}$ is continuous and $\tilde{a} < 0$ by Lemmas 2 and 3.

6 Examples

There is a useful characterization of a. Let ν denote the mean of H^+; and let

$$K(x) = \int_0^x \frac{1 - H^+(y)}{\nu} dy$$

for all $0 \le x < \infty$. If F is nonarithmetic then K is the asymptotic distribution of residual waiting time $S_{\tau_c} - c$ as $c \to \infty$. The following result does not require F to be nonarithmetic, however.

Theorem 3 *If $Kg(x) < \infty$, for all $0 \le x < \infty$, then $Kg(x)$ is minimized when $x = a$.*

Proof. It is clear that Kg is convex and, therefore, continuous. Using the assumption that Kg is finite, the monotonicity of g', and the Dominated Convergence Theorem, it is easily seen that

$$(Kg)'(x) = \int_0^\infty g'(x+y) \frac{1 - H^+(y)}{\nu} dy$$

for all $x \in \mathbb{R}$. Next, since g' is nondecreasing and H^+ has a finite mean, Fubini's Theorem leads to

$$\begin{aligned} \nu(Kg)'(x) &= \int_0^\infty \left[\int_y^\infty g'(x+y) H^+(dz) \right] dy \\ &= \int_0^\infty \left[\int_0^z g'(x+y) dy \right] H^+(dz) \\ &= \int_0^\infty [g(x+z) - g(x)] H^+(dz) \\ &= H^+g(x) - g(x) \end{aligned}$$

for all $x \in \mathbb{R}$. So, $(Kg)'(x)$ is nonpositive for $x < a$ and nonnegative for $x > a$. The theorem follows.

Example 1.

i) If $g(x) = |x - b|$ for all $x \in \mathbb{R}$, then $b - a$ must be a median of K.
ii) If $g(x) = (x - b)^2$ for all $x \in \mathbb{R}$ and H^+ has a finite second moment ν_2, then $b - a$ is the mean of K, $\nu_2/2\nu$.
iii) If $g(x) = e^{-x} + cx$ for all $x \in \mathbb{R}$, where $0 < c < 1$, then $b = log(1/c)$. If H^+ has a finite second moment and if

$$\kappa := \int_0^\infty e^{-x} K(dx)$$

then

$$Kg(x) = \kappa e^{-x} + c\left(x + \frac{\nu_2}{2\nu}\right)$$

is minimized when $x = log(\kappa/c)$.

Example 2. Suppose that $g(x) = e^{-x} + cx$ for all $x \in \boldsymbol{R}$,

i) If $\int_{\boldsymbol{R}} e^{-x} F(dx) = 1$, then $g(S_n)$, $n = 0, 1, 2, \ldots$, is a submartingale. So, the existence of a nontrivial optimal stopping time requires a violation of the hypotheses of the Optional Stopping Theorem.
ii) If F has Lebesgue density $f(x) = \frac{1}{3}e^{x}, -\infty < x < 0$ and $f(x) = \frac{2}{3}e^{-x}, 0 < x < \infty$, then H^+ is the standard exponential distribution and Theorem 2 is applicable, even though $E[g(S_n)] = \infty$ for all $n = 1, 2, \ldots$.

7 Proofs of Lemma 5 and (10)

Proof of Lemma 5. It is clear that $\rho(x) = E\{g(x+S_{\tau_{a-x}})\} = g(x) < \infty$, if $x > a$. So, it suffices to consider $x \leq a$. Let $c = a - x \geq 0$, and let U denote the renewal measure for the ladder height process, $U(B) = \sum_{k=0}^{\infty} H^{+*k}(B)$ for Borel sets $B \subseteq [0, \infty)$, where $*$ denotes convolution. Then, since $\{\tau_c = k\} = \{S_{\eta_{k-1}} \leq c\} \cap \{S_{\eta_k} > c\}$ for $k = 1, 2, \ldots$,

$$\begin{aligned} Eg(x + S_{\tau_c}) &= \int_0^c \left\{ \int_{c-y}^{\infty} g(x + y + z) H^+(dz) \right\} U(dy) \\ &\leq \int_0^c H^+ g(x + y) U(dy) \\ &\leq \int_0^c g(x + y) U(dy) \\ &< \infty, \end{aligned}$$

where the penultimate inequality follows from the definition of a.

Proof of (10). It is assumed throughout that $F(0) = 0$ and that $Fg(0) < \infty$. The details are supplied only for the case $x = 0$. There is no loss of generality in supposing that (2) holds, since values of $g(x)$ for $x < 0$ are irrelevant.

Let τ be a monotone case rule (for 0). Then $\tau \leq \tau_a$, so that $E[g(S_\tau)] < \infty$, by Lemma 5 and the convexity of g. Next, let t be another stopping time for which $E[g(S_t)] < \infty$; and let $A \in \mathcal{A}_{\tau \vee t}$. Then it suffices to show:

$$\int_{A, \tau = n < t} [g(S_t) - g(S_\tau)] dP = \int_{A, \tau = n < t} \left\{ \sum_{k=n}^{t-1} [Fg(S_k) - g(S_k)] \right\} dP \tag{14}$$

and

$$\int_{A, t = n < \tau} [g(S_\tau) - g(S_t)] dP = \int_{A, t = n < \tau} \left\{ \sum_{k=n}^{\tau-1} [Fg(S_k) - g(S_k)] \right\} dP \tag{15}$$

for all $n = 0, 1, 2, \ldots$, since (10) then follows by summing over n. The proofs of these two relations are similar. Only the first is given in detail. Let $m > n$ be an integer and write $t' = t \wedge m$. Then $g(S_{t'})$ is integrable, since $g(S_{t'}) \leq g(0) + g(S_t)$. Here

$g(S_k) - g(S_{k-1}) \geq g(X_k) - g(0)$ are bounded below by integrable random variables for all $k = 1, 2, \ldots$, by the convexity of g. So,

$$\int_{A,\tau=n<t} [g(S_{t'}) - g(S_\tau)]dP = \sum_{j=n+1}^{m} \int_{A,\tau=n,t'=j} \{ \sum_{k=n+1}^{j} [g(S_k) - g(S_{k-1})]\}dP$$
$$= \sum_{k=n+1}^{m} \int_{A,\tau=n,t'\geq k} [g(S_k) - g(S_{k-1})]dP.$$

Now, $A \cap \{\tau = n, t' \geq k\} = (A \cap \{\tau \vee t \geq k\}) \cap \{\tau = n\} \in \mathcal{A}_{k-1}$ for all $k = n+1, \ldots, m$. So, the last line is

$$\sum_{k=n+1}^{m} \int_{A,\tau=n,t'\geq k} [g(S_k) - g(S_{k-1})]dP$$
$$= \sum_{k=n+1}^{m} \int_{A,\tau=n,t'\geq k} [Fg(S_{k-1}) - g(S_{k-1})]dP$$
$$= \int_{A,\tau=n<t} \left\{ \sum_{k=n}^{t'-1} [Fg(S_k) - g(S_k)] \right\} dP.$$

Relation (13) now follows from Lemma 1 and the Monotone Convergence Theorem by letting $m \to \infty$.

References

Chow, Y. S., H. Robbins, and D. Siegmund (1971). *Great Expectations*. Houghton-Mifflin.

Feller, W. (1971). *An Introduction to Probability Theory and its Applications*, 2 Wiley.

Gut, A. (1988). *Stopped Random Walks*. Springer.

Lerche, H. R. (1991) Testing "$\theta = 0$" sequentially for normal random walks. Handout for talk at Oberwolfach.

Lorden, G. (1977). Nearly optimal tests for finitely many parameters. *Ann. Statist.* 5, 1–21.

Neveu, J. (1975). *Foundations of the Calculus of Probability*. Holden Day.

Woodroofe, M. (1982). *Nonlinear Renewal Theory in Sequential Analysis*. S.I.A.M.