FOURIER BASED METHODS FOR THE MANAGEMENT OF COMPLEX LIFE INSURANCE PRODUCTS

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Abstract. This paper proposes a framework for the valuation and the management of complex life insurance contracts, whose design can be described by a portfolio of embedded options, which are activated according to one or more triggering events. These events are in general monitored discretely over the life of the policy, due to the contract terms. Similar designs can also be found in other contexts, such as counterparty credit risk for example. The framework is based on Fourier transform methods as they allow to derive convenient closed analytical formulas for a broad spectrum of underlying dynamics. Multidimensionality issues generated by the discrete monitoring of the triggering events are dealt with efficiently designed Monte Carlo integration strategies. We illustrate the tractability of the proposed approach by means of a detailed study of ratchet variable annuities, which can be considered a prototypical example of these complex structured products.
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1. Introduction

In times of uncertainty, severe market fluctuations and very low interest rates such as the ones currently experienced by the large majority of the world economies, classical insurance contracts offering a guaranteed principal and a minimum interest rate are becoming less and less attractive. Indeed, the insurance industry focus has shifted to more sophisticated policies capable of providing higher returns to beat inflation, by giving the opportunity to participate in the growth of the economy. Examples of such contracts are represented by structured annuities such as fixed indexed annuities and registered index-linked annuities (RILA), which are a mix of standard fixed and variable annuities. RILAs in particular have been driving the 2019 annuity market to the highest annual sales since 2008, according to the Secure Retirement Institute.

This broad class of structured products presents a high degree of complexity as in general they offer opportunities for growth - up to a prespecified ceiling - based on the performance of a stock market index, paired with downside protection in the form of either a guaranteed floor or a buffer. Further, as insurance contracts, these products also offer protection against biometric risks, such as death, and the possibility of early termination of the policy (surrender).

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The payoff design of these products can be generally described by a portfolio of embedded options which are activated according to whether certain triggering events, such as death or surrender, occur. These events are usually discretely monitored by the issuers, and might be partially dependent on the sources of risk affecting the underlying of the embedded options, which implies that these policies are effectively path-dependent multi-asset derivatives. Similar schemes are also present in other contexts, such as counterparty credit risk for example, as relevant quantities such as Credit Value Adjustments (CVA) and Debt Value Adjustments (DVA) can be broadly described in similar terms, the positive exposure being the embedded option and the default of the counterparty representing the triggering event (see, for example, Ballotta et al., 2019, and references therein). Such sophisticated schemes call for equally sophisticated modelling approaches, advanced valuation techniques, and innovative numerical strategies for the management of the policies and the risks they entail.

In light of the above, the aim of this paper is to propose a realistic integrated model of the underlying risks affecting these complex structured products, in our specific case these being the financial risk (i.e. performance of the reference index) and the insurance risk (i.e. mortality and surrender), for the development of a market consistent framework for their valuation and management. We cast our stochastic model in a hybrid setting in order to accommodate for both the dependence between the equity and the fixed income market, and between the financial risks and surrender risk. In this context, Fourier transform methods are successfully employed for the pricing of such structured products, as explicit expressions for the valuation formulas can be obtained even in a fairly general setting as the one put forward in this paper. The key feature enabling this is the separation of the underlying stochastic dynamics and the payoff function, as highlighted in Eberlein et al. (2010).

When Fourier transform techniques are applied to path-dependent products such as the ones under consideration, a significant curse of dimensionality can be encountered due to the resulting high-dimensional Fourier integrals which need to be computed. The relevant dimension depends on the frequency with which the triggering event, i.e. early termination, is allowed based on the terms of the contract. From the computational point of view, we solve this dimensionality problem by means of Monte Carlo integration with importance sampling, as deterministic quadrature methods can in practice be implemented for at most 3 dimensions, which are generally inadequate for the typical maturities of VA contracts. For accurate valuation, care has to be paid to the choice of the importance sampling distribution, which is intrinsically linked to the shape of the surrender intensity function. Our choice is motivated by the fact that the rate of convergence of Monte Carlo integration is affected only by the number of iterations, rather than the dimensionality of the integral. Further numerical efficiency can be gained by interpolating some key quantities linked to the integrated cumulant functions of the driving processes.

We illustrate the methodology by considering ratchet variable annuities, which represent a prototypical example of the structured insurance products described above. In order to provide sufficient distributional flexibility to realistically capture the relevant dynamics, we employ time-inhomogeneous Lévy processes as in Ballotta et al. (2020) for the joint model of interest rate and equity risks, and insurance risk. In terms of financial modelling, this choice is consistent with the recent advances in the quantitative finance literature, which point towards stochastic models beyond Brownian motions. Further, such modelling for the financial market is particularly indicated in the current conditions: the large exposures of life insurance companies to movements in interest rates, due to their large holdings of fixed income securities, could depress significantly their solvency position, especially during a global economic recession such as the one expected post Covid–19 (see Yong, 2020, for example).

Further, we capture surrender risk via an intensity-based approach, which takes into account both the policyholder personal contingencies, and the spread between the return offered by the policy and the one offered on the market for equivalent products. This modelling approach recognizes that the decision of surrendering the insurance policy is not necessarily optimal from a strict financial point of view, as it could be impacted by personal contingencies as well. We note that for insurance companies offering variable annuities contracts, gaining insight into the surrender behaviour of policyholders is of particular importance because it can cause significant liquidity pressure.

To this purpose, we offer a general stochastic model which combines the two recognised theories put forward in the literature in order to explain early termination decisions, i.e. the Interest Rate Hypothesis and the Emergency Fund Hypothesis. The former links the reasons for surrendering to the evolution of the financial markets, so that a rational (in the financial sense) policyholder would actively switch to higher yield products; the latter suggests that surrender activity can increase during periods of economic duress as well. For example, it is perceived by the industry as well as by regulators that the income loss due to the current Covid–19 pandemia might alter the policyholder behaviour, and the corresponding surrender probabilities (Yong, 2020). We also observe that surrender modelling is particularly challenging due to lack of reliable data on actual termination causes, and consequent dependence on expert judgement. Therefore it is a potential source of risk which requires careful management.

Finally, mortality risk is modelled relying on an intensity-based approach, and independence of the financial risk under the risk neutral measure Q. In this context, we mention the contribution of Dhaene et al. (2013) showing that in general independence under the objective measure P - which is not relevant for our purposes - does not imply independence under Q. However, the QP-rule by Artzner et al. (2020), which is motivated by a fundamental theorem on insurance-finance arbitrage, allows to preserve this independence. See also Section 3.2 for a more detailed discussion.

By means of the developed computational procedures, we explore the role of the contract parameters in controlling for the risk/return tradeoff in the policy. This can prove useful to both the insurer and the policyholder in order to gain insight into the terms of the scheme.

We conclude this Introduction with a brief review of the literature closely related to this topic, which lies at the intersection of insurance, financial engineering and computational finance, highlighting the innovative points that we contribute.

The pricing of life insurance contracts in presence of financial risk has been studied quite extensively in the literature and traces back to Brennan and Schwartz (1976) and Boyle and Schwartz (1977). More recent contributions include Bacinello and Ortu (1996), Grosen and Jørgensen (2002), Bacinello et al. (2011), Deelstra and Rayée (2013), Giacinto et al. (2014) and Gudkov et al. (2019) amongst others. Bacinello et al. (2011) also offer an extensive overview of the benefits offered by Variable Annuities, such as the Guaranteed Minimum

Accumulation Benefit, the Guaranteed Minimum Death Benefit, and the Guaranteed Minimum Withdrawal Benefit, to mention a few. Ratchet designs have been discussed also in Lin and Tan (2003) and Kijima and Wong (2007). Modelling based on Lévy processes for the valuation of life insurance policies has been introduced by Ballotta (2005), and further explored in Ballotta (2009, 2010), Bacinello et al. (2016), Kélani and Quittard-Pinon (2017) and Alonso-Garcia et al. (2018) amongst others.

With respect to the above mentioned contributions, our model formulation distinguishes itself for the fact that the stochastic dependence between interest rate markets and stock markets is explicitly modelled in a non-diffusive setting accommodating time-varying volatility effects, following the hybrid construction of Eberlein and Rudmann (2018). These features are essential for long dated contracts such as the structured annuities under consideration.

Modelling mortality in a stochastic setting traces back to Milevsky and Promislow (2001) who adopt an intensity-based approach. Extensions have been proposed in the literature, and we cite, amongst many, the works of Dahl (2004), Biffis (2005) and Dahl and Møller (2006), who all work in the setting of affine diffusion processes. In this respect, we follow these indications and adopt an affine model based in which mortality risk is assumed independent of the financial risk, in line with the large majority of the literature.

As far as surrender modelling is concerned, contributions in the literature can be broadly classified in two categories. The first one tackles the surrender from the contingent claim point of view by valuing these early termination rights using the theory of American and Bermudan options (see, for example, Albizzati and Geman, 1994, Tanskanen and Lukkarinen, 2003, and the already mentioned Bacinello et al., 2011, 2016, Alonso-Garcia et al., 2018 and references therein). The second class of models, on the other hand, starts from the premise that the exercise of the surrender option is not necessarily the result of an optimal stopping problem, as the decision can be characterised by a degree of non-rationality due to personal motivations. Furthermore, it does not seem reasonable to expect that the large majority of annuity holders can closely monitor the financial market, and consider these life insurance policies as speculative instruments. Thus, the surrender decision is modelled as an exogenous factor via an intensity-based approach, which can be linked to both movements in the interest rate curve (Interest Rate Hypothesis), as well as non financial causes (Emergency Fund Hypothesis). Contributions in this class are the ones, for example, of Kolkiewicz and Tan (2006), Le Courtois and Nakagawa (2013), Loisel and Milhaud (2011) and Russo et al. (2017). This modelling choice is also consistent with market practice - see, for example, CEIOPS (2007) and Ducuroir et al. (2016).

Our contribution belongs to this latter stream of models and considers a stochastic intensity driven by more general processes, with explicit dependence on the dynamics of the financial markets, which attempts at unifying the arguments from both the Interest Rate Hypothesis and the Emergency Funds Hypothesis. Our formulation is also fairly general in terms of the shape of the intensity function, which leaves a certain degree of freedom for insurance companies to match their data and experience concerning early termination.

Transform methods have a long tradition in mathematics, and have become common in financial engineering. Some early contributions to the latter field include Carr and Madan (1999) and Raible (2000). A systematic study of the mathematical assumptions required for the use of the Fourier method is offered in Eberlein et al. (2010). The application of transform valuation approaches to different option contracts with and without path-dependence features has been studied in Cai and Kou (2012), Sesana et al. (2014), Cai et al. (2015), Fusai et al. (2016), Cui et al. (2017) to mention a few. For applications aimed at enhancing Monte Carlo simulation schemes, we refer to for example Broadie and Kaya (2006) and Cai et al. (2017).

We contribute to the discussion in this field by illustrating the application of Monte Carlo integration for the computation of the relevant high-dimensional Fourier integrals. Efficiency is gained by a suitable application of importance sampling on the one side, and interpolation for the reduction of the computational burden on the other side. We note that dimension reduction techniques for high-dimensional pricing integrals have been proposed in Wang (2006) and Wang and Tan (2013); however, these consider only dynamics driven by Brownian motions.

The paper is organized as follows. In section 2 we introduce the model for the financial market. Section 3 presents the contract features of the ratchet VA, and introduces the required modelling assumptions regarding mortality and surrender risk. We present the closed analytical formulas in section 4, and the numerical analysis in section 5. Section 6 concludes. Some additional material is offered in the Appendices.

2. The financial market

In this section we present the hybrid financial model for the joint dynamics of interest rates and equity prices. For this purpose, we consider two independent time-inhomogenous Lévy processes $L^1 = (L^1_s)_{s\geq 0}$ and $L^2 = (L^2_s)_{s\geq 0}$ - the interested reader can refer to Appendix A for the necessary definitions and assumptions.

Market incompleteness generated by such processes as drivers is addressed by standard practice of fixing the pricing measure via calibration, using market quotes for traded derivatives.

The model of the fixed income market is based on the approach in Eberlein et al. (2005) (see also Eberlein and Raible, 1999); thus, the relevant quantities of interest are the instantaneous forward rates $(f(t,T)_{0\leq t\leq T\leq T^*})$, which, for a deterministic and bounded function $f(0,T)$, are described by

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds - \int_0^t \sigma_1(s,T)dL^1_s. \tag{1}$$

The price $(S_t)_{0\leq t\leq T^*}$ of a generic equity index is, on the other hand, given by

$$S_t = S_0 \exp\Big(\int_0^t r(s)ds + \int_0^t \sigma_2(s)dL^2_s + \int_0^t \beta(s)dL^1_s - \omega(t)\Big); \tag{2}$$

the dependence with the dynamics of the forward rate is captured explicitly by $\beta(\cdot)$, and implicitly by the (integrated) short rate $r(t)$. This short rate is defined by equation (1) for $T = t$. Without loss of generality, we assume $S_0 = 1$.

The dynamics in equations (1) and (2) depend on $\sigma_1(\cdot,\cdot), \sigma_2(\cdot)$, $\beta(\cdot)$ which model the volatility of the quantities of interest, and are assumed to be continuous functions. Furthermore, the two dynamics depend on $\alpha(\cdot)$ and $\omega(\cdot)$ which ensure the martingale property for discounted asset prices. We discuss these quantities in more details in the following.

The volatility functions need to satisfy certain boundedness conditions in order to ensure the existence of exponential moments. Let M_1 and M_2 be the constant M from Assumption A.1 corresponding to the driving processes $L^1 = (L^1_s)_{s\geq 0}$ and $L^2 = (L^2_s)_{s\geq 0}$ respectively. Then, we assume that $\sigma_2(\cdot)$ is positive, and at the same time that $\sigma_2(\cdot)$ and $\beta(\cdot)$ satisfy

$$|\beta(s)| \leq \frac{M_1}{7}, \qquad \sigma_2(s) \leq \frac{M_2}{2}. \tag{3}$$

Concerning the conditions on $\sigma_1(\cdot,\cdot)$, we first need to define the price of a zero coupon bond with maturity T at time $t \leq T$ as $B(t,T) = \exp\left(-\int_t^T f(t,s)ds\right)$. Due to Fubini's theorem and equation (1), the bond price can be written as

$$B(t,T) = B(0,T)\exp\left(\int_0^t (f(s,s) - A(s,T))ds + \int_0^t \Sigma(s,T)dL^1_s\right),$$

with

$$A(s,t) := \int_s^t \alpha(s,u)du, \quad \Sigma(s,t) := \int_s^t \sigma_1(s,u)du.$$

Therefore, we assume $\sigma_1(\cdot,\cdot)$ such that

$$\Sigma(s,T) \leq \frac{M_1}{7}, \qquad 0 \leq s \leq T \tag{4}$$

to guarantee sufficient integrability in the proofs in the following sections.

Finally, in order to guarantee an arbitrage free market, we choose (using (A.2))

$$A(s,T) = \theta^1_s(\Sigma(s,T)), \quad 0 \leq s \leq T, \tag{5}$$

for the discounted bond prices to form Q-martingales; similarly, the discounted stock prices satisfy the martingale condition under Q if

$$\omega(t) = \int_0^t \left(\theta^2_s(\sigma_2(s)) + \theta^1_s(\beta(s))\right) ds, \tag{6}$$

which follows from (A.2) and the independence of the two driving processes $L^1 = (L^1_s)_{s\geq 0}$ and $L^2 = (L^2_s)_{s\geq 0}$.

This class of hybrid models has been studied in Eberlein and Rudmann (2018), who also provide the details of the calibration procedure.

3. Variable annuities: benefits and insurance modelling

3.1. **The VA contract.** A variable annuity (VA) is a type of annuity contract sold by insurance companies, the payoff of which can vary based on the performance of an underlying reference fund. VAs provide the policyholder a variety of benefits, including the so-called Guaranteed Minimum Accumulation Benefit (GMAB), and a Death Benefit (DB), which applies in case of early death of the policyholder.

A typical feature of these contracts is the periodic reset (ratchet), in the sense that the return to the policyholder is credited periodically based on a percentage of the realized performance of the underlying. Furthermore, VAs typically offer a downside protection to the policyholder by means of a guaranteed minimum rate; at the same time insurers might apply an upside limit in form of a cap to the credited return.

The contract can also include the right to cancel the policy before its contractual expiry date - the so-called surrender - and receive a pre-determined amount, which we refer to as the Surrender Benefit (SB). As surrenders represent a significant risk for the VA issuers, insurance companies usually impose penalties and charges aimed at reducing the amount available to the policyholder, making this option less appealing.

In the following we assume that the contract notional $I > 0$ is fully invested in the index with price process S_t. The maturity of the contract is set at $T \in (0, T^*]$, whilst we denote by $\tau^m(x)$ the remaining lifetime of an x years old policyholder, and by τ^s the random time of the policyholder surrender.

In details, define $\mathbf{t} := (t_0, t_1, \dots, t_N)^\top$ with $t_0 = 0$, and $t_N = T$; at maturity T the GMAB pays to the policyholder

$$Z_T := I\big(1 + \sum_{j=1}^{N} \max(\varphi, \min(\gamma, \xi R_{t_j}))\big);$$

$R_{t_j} := (S_{t_j} - S_{t_{j-1}})/S_{t_{j-1}}$ is the return during the period $[t_{j-1}, t_j]$ of the asset S_t, $j = 1, \dots, N$, the positive constants γ and φ are the cap and floor rates which refer to the ratchet periods $[t_{j-1}, t_j]$, $\xi \in (0, 1)$ is the participation rate. A close inspection of this payoff function reveals that the GMAB can be decomposed in a risk-free component represented by the guaranteed floor rate, and a sequence of call options on the return rate R_t

$$Z_T = I\left(1 + N\,\varphi + \sum_{j=1}^{N}\left(\xi R_{t_j} - \varphi\right)^+ - \sum_{j=1}^{N}\left(\xi R_{t_j} - \gamma\right)^+\right). \tag{7}$$

This payoff, however, can only be claimed if the policyholder is still alive at time T and did not exercise the surrender option before. Thus, the associated cash-flow at maturity is

$$\mathrm{GMAB}(T) = \mathbb{1}_{\{\tau^m(x) > T\}} \mathbb{1}_{\{\tau^s > T\}} Z_T. \tag{8}$$

The DB retains a similar payoff structure; however, as this is paid (to the beneficiaries) in case of early death of the policyholder, monitoring of this event is required. To this purpose, define $\tilde{\mathbf{t}} := (\tilde{t}_0, \tilde{t}_1, \dots, \tilde{t}_{N'})^\top$ with $0 = \tilde{t}_0 < \tilde{t}_1 < \ldots < \tilde{t}_{N'} = T$, and $\tilde{t}_i$ denoting the time points at which mortality is monitored by the insurer over the lifetime of the contract. Let us define $\ell(\tilde{t}_i) = \max\{j | t_j \le \tilde{t}_i\}$. Then, in case of death in the interval $[\tilde{t}_{i-1}, \tilde{t}_i[$, the DB pays

$$Z_{\tilde{t}_i} := I\big(1 + \sum_{j=1}^{\ell(\tilde{t}_i)} \max(\varphi, \min(\gamma, \xi R_{t_j})) + (\tilde{t}_i - \ell(\tilde{t}_i))\varphi\big),$$

for $\tilde{t}_i \in \tilde{\mathbf{t}}$, $i = 1, \dots, N'$. A decomposition in a risk-free part and a portfolio of options similar to the one noted in the GMAB applies also in this case. Consequently, the DB cash-flow is

$$\mathrm{DB}(\tilde{t}_i) = \mathbb{1}_{\{\tilde{t}_{i-1} \le \tau^m(x) < \tilde{t}_i\}} \mathbb{1}_{\{\tau^m(x) < \tau^s\}} Z_{\tilde{t}_i}, \tag{9}$$

as this is payable only in case of no early surrender.

Finally, in case of early surrender, the right of refund is restricted to the current value of the policyholder share in the underlying index reduced by a compulsory surrender penalty, as previously discussed (see also Bacinello et al., 2010, Le Courtois and Nakagawa, 2013, for similar designs). Thus, define $\bar{\mathbf{t}} := (\bar{t}_0, \bar{t}_1, \dots, \bar{t}_K)^\top$ with $0 = \bar{t}_0 < \bar{t}_1 < \ldots < \bar{t}_K < T$, and let us assume that premature surrender is possible only after the first year at any time point $\bar{t}_i \in \bar{\mathbf{t}}$ with $i = 1, \dots, K-1$, and $\bar{t}_1 = 1$ year. Then the policyholder receives the amount

$$I S_{\bar{t}_i} P(\bar{t}_i),$$

with $P(\bar{t}_i)$ denoting the proportion of the current index value that the policyholder is entitled to, so that $1 - P(\bar{t}_i)$ amounts to the actual surrender penalty. In particular, we model this proportion P via an increasing function $P : [0,T] \to (0,1]$ such that $P(T) = 1$. As the surrender option can be exercised only if the policyholder is still alive (i.e., $\{\tau^s < \tau^m(x)\}$), should surrender occur at time $\bar{t}_i$, the corresponding cash-flow is

$$\mathrm{SB}(\bar{t}_i) = \mathbb{1}_{\{\tau^s = \bar{t}_i\}} \mathbb{1}_{\{\tau^s < \tau^m(x)\}} I S_{\bar{t}_i} P(\bar{t}_i). \tag{10}$$

We conclude the description of the VA contract by specifying the hierarchy of the three time scales $\mathbf{t}$, $\bar{\mathbf{t}}$, and $\tilde{\mathbf{t}}$. First we assume that $\bar{\mathbf{t}} \subset \tilde{\mathbf{t}}$, so that any surrender time $\bar{t}_i$ is contained in $\{\tilde{t}_1, \ldots, \tilde{t}_{N'}\}$. This is a very natural assumption since death of the policyholder might be monitored by the insurer at the end of every month or every quarter, whereas surrender of the contract might be allowed only at the end of each year, or with the exception of year 1 each half year, during the life of the contract.

Concerning $\mathbf{t}$, the typical contract design provides for annual ratcheting which would mean $t_i = i$. However, shorter ratchet periods such as crediting of the performance of the reference fund every six months could be of interest as well. Therefore, we assume only that with the exception of possible time points t_i before the end of the first year - recall that surrender is not allowed during the first year - all time points t_i are contained in $\bar{\mathbf{t}}$.

In terms of valuation, the payoff functions (8) – (10) highlight the need for a model not only for the financial market, but also for the random quantities τ^m and τ^s capturing mortality and surrender risk respectively. The modelling of these insurance risks is covered in the following sections.

3.2. **Modelling of mortality risk.** As mentioned in the Introduction, we adopt a stochastic intensity-based approach for mortality, obtained by superimposing a given initial curve for mortality rates and a stochastic process modelling the random improvements and fluctuations of mortality trends.

To this purpose, as $\tau^m(x)$ denotes the random time describing the remaining lifetime of an x years old individual, the survival probability is

$$Q\left(\tau^m(x) > t\right) = \mathbb{E}_Q\left(e^{-\int_0^t \lambda_u^m(x+u)du}\right). \tag{11}$$

In the above, $\lambda_t^m(x+t)$, $t > 0$, denotes the stochastic intensity associated to τ^m for an individual aged $x+t$ at time t. We define it as

$$\lambda_t^m(x+t) = \lambda^{m,0}(x+t)\xi_t, \tag{12}$$

with $\lambda_0^m(x) = \lambda^{m,0}(x)$, for an initial curve of the mortality intensity, $\lambda^{m,0}$, and a process ξ_t such that $\xi_0 = 1$, capturing the mortality improvements from time 0 to time t for a person aged $x+t$.

The initial curve is assumed to follow the standard Gompertz-Makeham model, that is

$$\lambda^{m,0}(x+t) = \frac{1}{q} e^{\left(\frac{x+t-z}{q}\right)}, \quad z \geq 0, q > 0.$$

The process for the mortality improvement ratio ξ_t follows a generalized Ornstein-Uhlenbeck process of the form

$$d\xi_t = \kappa(\exp(-\lambda t) - \xi_t)dt + dW_t, \quad \lambda \in \mathbb{R}$$

for $W = (W_s)_{s\geq 0}$ a Brownian motion independent of (L^1, L^2). It follows that for any $0 \leq t \leq T$

$$Q(\tau^m(x) > t) = \exp\Big(A_x(t) + B_x(t)\lambda_0^m(x)\Big); \tag{13}$$

$A_x(t), B_x(t)$ are functions of the parameters of the models for $\lambda^{m,0}(x+t)$ and ξ_t; their explicit expressions are provided in the (online) Appendix D.

A few considerations are in order. The adopted specification of the process ξ_t does not guarantee that the mortality intensity is positive; however as noted already in Escobar et al. (2016), for the chosen parameters (see Section 5) the probability of this event is of order 10^{-5} and therefore negligible. Furthermore, one possible generalization could replace the Brownian motion W by a positive Lévy process $L^3 = (L^3_s)_{s\geq 0}$ independent of (L^1, L^2). The survival probability (11) can then be derived using the theory of affine processes as described, for example, in Eberlein and Kallsen (2019), Chapter 6. Another possible generalization could be the extension to multi-factor mortality models. We leave these to further research.

Furthermore, in this construction we assume independence between demographic and financial risks under the risk-neutral measure Q. In this regard, we note the following. There is an agreement in the actuarial literature that independence between mortality risk and financial risks under the objective measure P is a reasonable assumption in many cases. However, a change to an equivalent martingale measure, such as the pricing measure Q, could introduce dependence under this martingale measure, see Dhaene et al. (2013) for a detailed discussion. On the other hand, the QP-rule by Artzner et al. (2020), which is motivated by a fundamental theorem on insurance-finance arbitrage, suggests keeping as many properties of the objective measure as possible. In particular, the QP-rule implies that independence between the biometric and the financial risks translates into independence also under the equivalent measure used for the valuation of life insurance products. We conclude by also noting that many other equivalent measures have this property, although not all of them.

3.3. **Modelling of surrender risk.** In the following we adopt an intensity-based approach to model the policyholder surrender behaviour. Specifically, we assume that the policyholder's decision to end prematurely the insurance contract is affected by personal and financial considerations, although this decision might not be necessarily optimal from the point of view of financial theory.

The surrender intensity is defined by two components, a deterministic baseline given by a constant $C > 0$, and a dynamic component. The baseline captures surrender behaviours induced by personal contingencies and any other non-economic cause. For the definition of the dynamic part, we consider a process $D(t)$ defined as

$$D(t) = Y_t - p(t) + \int_t^T f(t,s)ds - \delta T, \quad 0 \leq t \leq T, \tag{14}$$

capturing the spread between the return guaranteed to the policyholder at maturity in case of no surrender on the one hand, and the surrender benefit plus any return offered by reinvesting in the fixed income market on the other hand. In more details, in (14), the guaranteed return in case of no surrender is represented by δT, with $\delta = \log(1 + N\varphi)/T$, i.e. the continuously compounded rate equivalent to the floor rate φ accumulated up to maturity T. The return in case of surrender consists of the benefit at the time of surrender, which is determined by the (log-)return of the equity index, $Y_t = \log S_t$, net of any charges and penalties (captured by the term $p(t) = -\log P(t))$, plus the rates at which this amount can be reinvested in the fixed income market until maturity T.

As specified above, we assume that surrender is allowed at time points $\bar{t}_i$, $i = 1, \ldots, K-1$. Based on the definitions of C and $D(t)$, the surrender intensity λ^s is constructed as a variable which is constant on each interval $[\bar{t}_i, \bar{t}_{i+1})$ for $i = 1, \ldots, K-1$. Specifically, we set

$$\lambda^s(t) = \beta \ h(D(\bar{t}_i)) + C, \quad \text{for } \bar{t}_i \leq t < \bar{t}_{i+1}, \tag{15}$$

and $\lambda^s(t) = 0$ for $t \in [0, \bar{t}_1) \cup [\bar{t}_K, T]$. The parameter $\beta \in [0,1]$ measures the extent of the impact of the conditions in the financial market on the policyholder decision to leave the VA scheme. The function $h(\cdot)$ is assumed non-negative and continuous, and satisfying additional integrability conditions in the following sense. We define for $l = 1, \ldots, K-1$ and $\Delta\bar{t}_l := \bar{t}_l - \bar{t}_{l-1}$ the function

$$f_l(x) = e^{-\beta \Delta \bar{t}_{l+1} h(x)},$$

and require the integrability of both f_l and $\hat{f}_l$, with $\hat{f}$ denoting the Fourier transform of a generic function f. Possible examples are $h(x) = x^2$ and $h(x) = |x|$.

Finally, the conditional probability that surrender does not occur until $\bar{t}_i$ is

$$Q(\tau^s \geq \bar{t}_i | \mathcal{F}_{\bar{t}_i}^{L^1,L^2}) = \exp\Big(- \int_0^{\bar{t}_i} \lambda^s(u) du \Big), \tag{16}$$

for all $1 \leq i \leq K-1$, and

$$\begin{aligned} Q(\tau^s \geq t | \mathcal{F}_t^{L^1,L^2}) &= Q(\tau^s = \infty | \mathcal{F}_t^{L^1,L^2}) \\ &= \exp\Big(- \int_0^{\bar{t}_K} \lambda^s(u) du \Big), \end{aligned} \tag{17}$$

for $\bar{t}_{K-1} < t$. Here $\left(\mathcal{F}_t^{L^1,L^2}\right)_{t \geq 0}$ denotes the filtration which is generated by the driving processes $(L_t^1)_{t \geq 0}$ and $(L_t^2)_{t \geq 0}$. Note that the right hand side of Equation (17) is measurable with respect to $\mathcal{F}_{\bar{t}_{K-1}}^{L^1,L^2}$ due to the definition of $\lambda^s(t)$ on the interval $[\bar{t}_{K-1}, \bar{t}_K)$.

At this stage, we observe that an important decision in the modelling process concerns the choice of the function $h(x)$, as this influences the reaction of the surrender model to the changes in the current market conditions. Although the approach offered in this paper is quite general, we consider the case of non-negative functions $h(\cdot)$, as the above mentioned parabolic and modulus functions, as a way of combining together the two recognized theories in the literature attempting to explain the surrender behaviour, i.e. the Interest Rate Theory and the Emergency Fund Theory (see for example Knoller et al., 2016).

In short, according to the former theory, both higher and lower interest rates (such as the ones currently observed in all major economies) can represent strong incentives for the policyholder to surrender in order to be able to either switch to higher yield investments, or to exploit advantageous refinancing opportunities respectively. The Emergency Fund Theory, on the other hand, puts forward the idea of surrender as a way to regaining sufficient financial resources following individual income shocks.

A few final considerations are in order. Firstly, consistently with the literature in the field (see for example Milhaud et al., 2011, Knoller et al., 2016, Nolte and Schneider, 2017, and references therein), the intensity functions λ^s and λ^m are independent due to the assumed independence between demographic and financial risks. Secondly, our construction differs from the ones put forward by Le Courtois and Nakagawa (2013), Escobar et al. (2016) and

Russo et al. (2017) for example, due to the non-Gaussian dynamics of the risk drivers L^1 and L^2. Finally, our approach generalizes Ballotta et al. (2020) as the intensity function λ^s uses a generic non-negative function $h(\cdot)$.

4. Fourier-based market consistent valuation

In this section we derive closed form valuation formulas for the VA components introduced in Section 3 based on the theory of Fourier transforms.

On the basis of the risk-neutral valuation principle, the value at time $t = 0$ of each contract component is

$$\mathrm{P}^{\mathrm{GMAB}} = E_Q\left[e^{-\int_0^T r(u)du}\,\mathrm{GMAB}(T)\right], \quad \mathrm{P}^{\mathrm{DB}} = \sum_{i=1}^{N'} E_Q\left[e^{-\int_0^{\tilde{t}_i} r(u)du}\mathrm{DB}(\tilde{t}_i)\right]$$

and

$$\mathrm{P}^{\mathrm{SB}} = \sum_{i=1}^{K-1} E_Q\left[e^{-\int_0^{\bar{t}_i} r(u)du}\mathrm{SB}(\bar{t}_i)\right].$$

As the value P^{VA} of the variable annuity is the sum of the values of its constituents,

$$\mathrm{P}^{\mathrm{VA}} = \mathrm{P}^{\mathrm{GMAB}} + \mathrm{P}^{\mathrm{DB}} + \mathrm{P}^{\mathrm{SB}}. \tag{18}$$

We mention that in the literature on valuation of life insurance contracts, fees aimed at covering the cost of the guarantees and other management expenses are considered as well, see for example Bacinello et al. (2011), Kélani and Quittard-Pinon (2017) and references therein. However, in the following we focus instead on risk-neutral valuation. The proposed model, though, can be adapted to cater for these additional features by introducing a fee rate applied to the fund value.

4.1. **Guaranteed minimum accumulation benefit.** We provide in Theorem 4.1 an explicit representation for the value $\mathrm{P}^{\mathrm{GMAB}}$, which is numerically efficient as it reduces the task to the computation of two integrals. In the interest of readability, the corresponding integrand functions M and N are defined in Appendix B.1.

Theorem 4.1. *The value* $\mathrm{P}^{\mathrm{GMAB}}$ *is given by*

$$\mathrm{P}^{\mathrm{GMAB}} = Q(\tau^m(x) > T)B(0,T)I\left((1+N\varphi)A_1 + \xi\sum_{j=1}^{N}\left(A_{2,j}\left(\frac{\varphi}{\xi}\right) - A_{2,j}\left(\frac{\gamma}{\xi}\right)\right)\right)$$

with

$$A_1 = \frac{e^{-C(\bar{t}_K-\bar{t}_1)}}{(2\pi)^{K-1}}e^{-\int_0^T A(s,T)ds}\int_{\mathbb{R}^{K-1}} M(u,T)du,$$

$$A_{2,j}(\kappa) = \frac{e^{-C(\bar{t}_K-\bar{t}_1)}}{(2\pi)^{K}}e^{-\int_0^T A(s,T)ds}\int_{\mathbb{R}^{K}} N(u,j,\kappa,T)du,$$

and $A(s,T)$ *as in* (5).

Remark 4.1 (Some preliminaries)**.** We note that in the given setting the following result holds in virtue of the explicit formula for the bond price $B(t,T)$

$$-\int_t^T f(t,s)ds = \int_0^t r(s)ds - \int_0^T f(0,s)ds - \int_0^t A(u,T)du + \int_0^t \Sigma(u,T)dL_u^1. \tag{19}$$

In addition, by setting $t = T$ in (19), we obtain

$$0 = \int_0^t r(s)ds - \int_0^t f(0,s)ds - \int_0^t A(s,t)ds + \int_0^t \Sigma(s,t)dL_s^1; \tag{20}$$

Further, equation (19) together with equation (2) allows the following explicit representation of $D(t)$

$$\begin{aligned} D(t) &= \int_0^t r(s)ds + \int_0^t \sigma_2(s)dL_s^2 + \int_0^t \beta(s)dL_s^1 - \omega(t) - p(t) + \int_t^T f(t,s)ds - \delta T \\ &= -p(t) - \delta T + \int_0^T f(0,s)ds + \int_0^t A(s,T)ds - \int_0^t \Sigma(s,T)dL_s^1 \\ &\quad + \int_0^t \sigma_2(s)dL_s^2 + \int_0^t \beta(s)dL_s^1 - \omega(t). \end{aligned} \tag{21}$$

Proof of Theorem 4.1. Our starting point is the payoff function (8). The independence between $\tau^m(x)$ and the financial market implies that

$$\begin{aligned} \mathrm{P}^{\mathrm{GMAB}} &= E_Q\left[e^{-\int_0^T r(u)du}\mathbb{1}_{\{\tau^m(x)>T\}}\mathbb{1}_{\{\tau^s>T\}}\, Z_T\right] \\ &= Q(\tau^m(x)>T)E_Q\left[e^{-\int_0^T r(u)du} Z_T E_Q\left(\mathbb{1}_{\{\tau^s>T\}}|\mathscr{F}_T^{L^1,L^2}\right)\right]. \end{aligned}$$

By (17)

$$\mathrm{P}^{\mathrm{GMAB}} = Q(\tau^m(x)>T)E_Q\left[e^{-\int_0^T r(u)du}e^{-\int_0^{\bar{t}_K}\lambda^s(u)du}\, Z_T\right].$$

Let Q^T be the T-forward measure, i.e.

$$\frac{dQ^T}{dQ} = \frac{1}{B(0,T)B(T)}, \tag{22}$$

with $B(t) = \exp\left(\int_0^t r(s)ds\right)$, and E^T the corresponding expectation. Then

$$\mathrm{P}^{\mathrm{GMAB}} = Q(\tau^m(x)>T)B(0,T)E^T\left[e^{-\int_0^{\bar{t}_K}\lambda^s(u)du}\, Z_T\right].$$

It follows from (7) that

$$\begin{aligned} E^T\left[e^{-\int_0^{\bar{t}_K}\lambda^s(u)du}\, Z_T\right] &= I(1+N\varphi)E^T\left[e^{-\int_0^{\bar{t}_K}\lambda^s(u)du}\right] \\ &\quad + I\xi\sum_{j=1}^N E^T\left[e^{-\int_0^{\bar{t}_K}\lambda^s(u)du}\left(R_{t_j}-\frac{\varphi}{\xi}\right)^+\right] \\ &\quad - I\xi\sum_{j=1}^N E^T\left[e^{-\int_0^{\bar{t}_K}\lambda^s(u)du}\left(R_{t_j}-\frac{\gamma}{\xi}\right)^+\right] \\ &=: I(1+N\varphi)A_1 + I\xi\sum_{j=1}^N\left(A_{2,j}\left(\frac{\varphi}{\xi}\right) - A_{2,j}\left(\frac{\gamma}{\xi}\right)\right), \end{aligned} \tag{23}$$

with obvious definitions in the last line. As λ^s is a step function (see (15)), it follows that

$$\begin{aligned} A_1 &= e^{-C(\bar{t}_K-\bar{t}_1)}E^T\Big[\prod_{i=1}^{K-1} e^{-\beta\Delta\bar{t}_{i+1}h(D(\bar{t}_i))}\Big] \\ &= e^{-C(\bar{t}_K-\bar{t}_1)}E^T\Big[f^{K-1}(D(\bar{t}_1),\dots,D(\bar{t}_{K-1}))\Big], \end{aligned}$$

with $f^{K-1}(x_1,\dots,x_{K-1}) := \prod_{l=1}^{K-1} f_l(x_l)$ as defined in (B.1). As

$$\widehat{f^{K-1}}(y_1,\dots,y_{K-1}) = \prod_{l=1}^{K-1}\hat{f}_l(y_l) \tag{24}$$

and $\hat{f}_l$ is integrable by assumption, we conclude that $\widehat{f^{K-1}}$ as a function on $\mathbb{R}^{K-1}$ is integrable which is a necessary requirements for its inversion. By Theorem 3.2 (and Remark 3.1) in Eberlein et al. (2010) - note that dampening of the function is not required in this case -

$$E^T\Big[f^{K-1}(D(\bar{t}_1),\dots,D(\bar{t}_{K-1}))\Big] = \frac{1}{(2\pi)^{K-1}}\int_{\mathbb{R}^{K-1}}\tilde{M}(\mathrm{i}u)\widehat{f^{K-1}}(-u)du, \tag{25}$$

with

$$\tilde{M}(\mathrm{i}u) := E^T\Big[e^{\mathrm{i}u_1 D(\bar{t}_1)+\dots+\mathrm{i}u_{K-1}D(\bar{t}_{K-1})}\Big],$$

for any $u=(u_1,\dots,u_{K-1})$. Representation (21) above and the definition of w_l in equation (B.2) imply

$$\tilde{M}(\mathrm{i}u) = \exp\Big(\mathrm{i}\sum_{l=1}^{K-1}u_l w_l\Big)E^T\Big[\exp\Big(\mathrm{i}\sum_{l=1}^{K-1}u_l\Big(\int_0^{\bar{t}_l}\sigma_2(s)dL_s^2+\int_0^{\bar{t}_l}(\beta(s)-\Sigma(s,T))dL_s^1\Big)\Big)\Big]. \tag{26}$$

As

$$B(t) = \frac{1}{B(0,t)}\exp\Big(\int_0^t A(s,t)ds - \int_0^t \Sigma(s,t)dL_s^1\Big). \tag{27}$$

(see (20)), the explicit form of the density of Q^T is

$$\frac{dQ^T}{dQ} = \exp\Big(-\int_0^T A(s,T)ds + \int_0^T \Sigma(s,T)dL_s^1\Big).$$

It follows that the expectation in (26) can be written as

$$\begin{aligned} & E_Q\Big[\exp\Big(\int_0^T \Sigma(s,T))dL_s^1 - \int_0^T A(s,T)ds\Big) \\ &\qquad \times\exp\Big(\mathrm{i}\sum_{l=1}^{K-1}\Big(\int_0^{\bar{t}_l}u_l\sigma_2(s)dL_s^2+\int_0^{\bar{t}_l}u_l(\beta(s)-\Sigma(s,T))dL_s^1\Big)\Big)\Big] \\ &= e^{-\int_0^T A(s,T)ds}E_Q\Big[\exp\Big(\int_0^T E(s,u,T)dL_s^1+\int_0^T F(s,u)dL_s^2\Big)\Big], \end{aligned}$$

with E and F as in (B.4). As the two driving processes are independent, and the integrands are left-continuous functions, the expression above is nothing but

$$\exp\Big(-\int_0^T A(s,T)ds + \int_0^T \theta_s^1(E(s,u,T))ds + \int_0^T \theta_s^2(F(s,u))ds\Big)$$

in virtue of equation (A.2). With $D(u,T)$ defined in (B.4), $\tilde{M}(\mathrm{i}u)$ in (26) is

$$\tilde{M}(\mathrm{i}u) \quad = \quad D(u,T)\exp\Big(-\int_0^T A(s,T)ds + \int_0^T \theta_s^1(E(s,u,T))ds + \int_0^T \theta_s^2(F(s,u))ds\Big).$$

By plugging this representation of $\tilde{M}(\mathrm{i}u)$ in (25) the result for A_1 follows.

We turn our attention to the term

$$A_{2,j}(\kappa) = E^T\left[e^{-\int_0^{\bar{t}_K}\lambda^s(u)du}\Big(R_{t_j}-\kappa\Big)^+\right]. \tag{28}$$

As $R_{t_j} = \exp(Y_{t_j} - Y_{t_{j-1}}) - 1$, the same arguments as above lead to

$$\begin{aligned} A_{2,j}(\kappa) &= e^{-C(\bar{t}_K-\bar{t}_1)}E^T\left[f^{K-1}(D(\bar{t}_1),\ldots,D(\bar{t}_{K-1}))\left(e^{Y_{t_j}-Y_{t_{j-1}}}-1-\kappa\right)^+\right] \\ &= e^{-C(\bar{t}_K-\bar{t}_1)}E^T\left[F(D(\bar{t}_1),\ldots,D(\bar{t}_{K-1}),Y_{t_j}-Y_{t_{j-1}})\right], \end{aligned}$$

for

$$F(x_1,\ldots,x_K) := f^{K-1}(x_1,\ldots,x_{K-1})\,(e^{x_K}-1-\kappa)^+\,.$$

We note that the second factor of F is unbounded for large x_K; therefore, integrability has to be recovered via dampening. To this purpose, define $g(x_1,\ldots,x_K) := F(x_1,\ldots,x_K)e^{-rx_K}$, with $1<r<2$; further, let

$$g_K(x_K) := (e^{x_K}-1-\kappa)^+\,e^{-rx_K},$$

then, g_K and consequently also g are integrable functions. The Fourier transform of g_K is

$$\hat{g}_K(y) = \frac{(1+\kappa)e^{(\mathrm{i}y-r)\log(1+\kappa)}}{(\mathrm{i}y-r+1)(\mathrm{i}y-r)} = \frac{(1+\kappa)^{\mathrm{i}y-r+1}}{(\mathrm{i}y-r+1)(\mathrm{i}y-r)}, \quad y\in\mathbb{R}.$$

$\hat{g}_K$ is integrable as $|\hat{g}_K(y)|_{\mathbb{C}} = (1+\kappa)e^{-r\log(1+\kappa)}(((1-r)^2+y^2)(r^2+y^2))^{-1/2}$. It follows that

$$\hat{g}(y_1,\ldots,y_K) = \widehat{f^{K-1}}(y_1,\ldots,y_{K-1})\frac{(1+\kappa)^{\mathrm{i}y_K-r+1}}{(\mathrm{i}y_K-r+1)(\mathrm{i}y_K-r)}, \tag{29}$$

and therefore $\hat{g}$ is integrable. Theorem 3.2 in Eberlein et al. (2010) can again be applied with $R := (0,\ldots,0,r)\in\mathbb{R}^K$, and it allows to express the expectation above as a Fourier integral in $\mathbb{R}^K$, i.e.

$$E^T\left[F(D(\bar{t}_1),\ldots,D(\bar{t}_{K-1}),Y_{t_j}-Y_{t_{j-1}})\right] = \frac{1}{(2\pi)^K}\int_{\mathbb{R}^K}\tilde{N}_j(R+\mathrm{i}u)\hat{F}(\mathrm{i}R-u)du, \tag{30}$$

with $\tilde{N}_j(R+\mathrm{i}u)$ defined as

$$\tilde{N}_j(R+\mathrm{i}u) \quad := \quad E^T\left[e^{\mathrm{i}u_1D(\bar{t}_1)+\ldots+\mathrm{i}u_{K-1}D(\bar{t}_{K-1})+(\mathrm{i}u_K+r)(Y_{t_j}-Y_{t_{j-1}})}\right].$$

Using (20), the increments of Y_t can be written in the form

$$\begin{aligned} Y_{t_j}-Y_{t_{j-1}} \quad &= \quad \int_{]t_{j-1},t_j]} r(s)ds + \int_{]t_{j-1},t_j]}\sigma_2(s)dL_s^2 + \int_{]t_{j-1},t_j]}\beta(s)dL_s^1 - (\omega(t_j)-\omega(t_{j-1})) \\ &= \quad \widetilde{w}_{t_j} - \int_0^{t_j}\Sigma(s,t_j)dL_s^1 + \int_0^{t_{j-1}}\Sigma(s,t_{j-1})dL_s^1 \\ &\qquad + \int_{]t_{j-1},t_j]}\sigma_2(s)dL_s^2 + \int_{]t_{j-1},t_j]}\beta(s)dL_s^1. \end{aligned}$$

Thus, as above, using the notation from (B.4), we derive by switching back to the original risk neutral measure

$$\begin{aligned}
\tilde{N}_j(R+\mathrm{i}u) &= E_Q\Big[e^{\mathrm{i}u_1 D(\bar{t}_1)+\ldots+\mathrm{i}u_{K-1}D(\bar{t}_{K-1})+(\mathrm{i}u_K+r)(Y_{t_j}-Y_{t_{j-1}})-\int_0^T A(s,T)ds+\int_0^T \Sigma(s,T))dL_s^1}\Big] \\
&= \tilde{D}(u-\mathrm{i}R,j,T)e^{-\int_0^T A(s,T)ds} \\
&\times\ E_Q\Big[\exp\Big(\int_0^T \tilde{E}(s,u-\mathrm{i}R,j,T)dL_s^1+\int_0^T \tilde{F}(s,u-\mathrm{i}R,j)dL_s^2\Big)\Big].
\end{aligned}$$

Observe that as $1<r<2$, and in virtue of (3) and (4), $r\sigma_2(s)\le M_2$ and $|r\beta(s)-r\Sigma(s,t_j)+r\Sigma(s,t_{j-1})+\Sigma(s,T)|\le 6M_1/7+M_1/7=M_1$. This implies that the real part of the integrand in both stochastic integrals is bounded by M_1 and M_2 respectively. Consequently, by independence of the driving processes and the left-continuity of the integrands, the last expectation can be conveniently computed according to (A.2). The result is

$$\begin{aligned}
\tilde{N}_j(R+\mathrm{i}u) &= \tilde{D}(u-\mathrm{i}R,j,T)e^{-\int_0^T A(s,T)ds} \\
&\quad\times\exp\Big(\int_0^T \theta_s^1(\tilde{E}(s,u-\mathrm{i}R,j,T))ds+\int_0^T \theta_s^2(\tilde{F}(s,u-\mathrm{i}R,j))ds\Big). \quad (31)
\end{aligned}$$

$\hat{g}$ and $\hat{F}$ are related by

$$\hat{g}(u)=\int_{\mathbb{R}^K} e^{\mathrm{i}\langle u,x\rangle}e^{-\langle R,x\rangle}F(x)dx=\hat{F}(u+\mathrm{i}R),$$

and therefore

$$\hat{F}(\mathrm{i}R-u)=\hat{g}(-u) = \widehat{f^{K-1}}(-u_1,\ldots,-u_{K-1})\frac{(1+\kappa)^{1-\mathrm{i}u_K-r}}{(\mathrm{i}u_K+r-1)(\mathrm{i}u_K+r)}. \quad (32)$$

The result follows by substitution of (31) and (32) into (30). □

4.2. **Death benefit.** Similarly to the GMAB case, we provide a numerically efficient representation for the value of the death benefit

$$\mathrm{P}^{\mathrm{DB}}=\sum_{i=1}^{N'} E_Q\Big[e^{-\int_0^{\tilde{t}_i} r(u)du}\mathrm{DB}(\tilde{t}_i)\Big].$$

The main result is offered in the following theorem (in the interest of readability, the corresponding integrand functions N_0, $M^{j,i}$ and $N^{j,i}$ are defined in Appendix B.2).

Theorem 4.2.

$$\begin{aligned}
\mathrm{P}^{\mathrm{DB}} &= I\sum_{i:\,\tilde{t}_i\le\bar{t}_1} Q(\tau^m(x)\in[\tilde{t}_{i-1},\tilde{t}_i))B(0,\tilde{t}_i)\left((1+\tilde{t}_i\varphi)+\xi\sum_{l'=1}^{\ell(\tilde{t}_i)}\left(A_{i,l'}\left(\frac{\varphi}{\xi}\right)-A_{i,l'}\left(\frac{\gamma}{\xi}\right)\right)\right) \\
&+I\sum_{j=1}^{K-2}\sum_{i:\,\tilde{t}_i\in(\bar{t}_j,\bar{t}_{j+1}]} Q\left(\tau^m(x)\in[\tilde{t}_{i-1},\tilde{t}_i)\right)B(0,\tilde{t}_i)\left((1+\tilde{t}_i\varphi)A_{j,i}^1+\xi\sum_{l'=1}^{\ell(\tilde{t}_i)}\left(A_{j,i,l'}^2\left(\frac{\varphi}{\xi}\right)-A_{j,i,l'}^2\left(\frac{\gamma}{\xi}\right)\right)\right) \\
&+I\sum_{i:\tilde{t}_i\in(\bar{t}_{K-1},T]} Q(\tau^m(x)\in[\tilde{t}_{i-1},\tilde{t}_i))B(0,\tilde{t}_i)\left((1+\tilde{t}_i\varphi)A_{K-1,i}^1+\xi\sum_{l'=1}^{\ell(\tilde{t}_i)}\left(A_{K-1,i,l'}^2\left(\frac{\varphi}{\xi}\right)-A_{K-1,i,l'}^2\left(\frac{\gamma}{\xi}\right)\right)\right),
\end{aligned}$$

with

$$
\begin{aligned}
A_{i,l'}(\kappa) &= \frac{e^{-\int_0^{\tilde{t}_i} A(s,\tilde{t}_i)ds}}{2\pi} \int_{\mathbb{R}} N_0(u,l',\kappa,\tilde{t}_i)du,\\
A^1_{j,i} &= \frac{e^{-C(\bar{t}_{j+1}-\bar{t}_1)}}{(2\pi)^j} e^{-\int_0^{\tilde{t}_i} A(s,\tilde{t}_i)ds} \int_{\mathbb{R}^j} M^{j,i}(u,T)du,\\
A^2_{j,i,l'}(\kappa) &= \frac{e^{-C(\bar{t}_{j+1}-\bar{t}_1)}}{(2\pi)^{j+1}} e^{-\int_0^{\tilde{t}_i} A(s,\tilde{t}_i)ds} \int_{\mathbb{R}^{j+1}} N^{j,i}(u,l',\kappa,T)du,
\end{aligned}
$$

for $j \in \{1,\dots,K-1\}$.

The proof of Theorem 4.2 is built along similar arguments as the ones used for the proof of Theorem 4.1 and is, therefore, deferred to (the online) Appendix C.

4.3. **Surrender benefit.** We conclude by focusing on the valuation of the third component of the VA contract, namely the surrender benefit

$$
\mathrm{P}^{\mathrm{SB}} = \sum_{i=1}^{K-1} E_Q\Big[e^{-\int_0^{\bar{t}_i} r(u)du}\mathrm{SB}(\bar{t}_i)\Big].
$$

As in the previous sections, the definition of the integrand functions (M^i and N^i) in the representation offered in the following theorem is deferred to Appendix B.3.

Theorem 4.3. *The value* P^{SB} *has the explicit representation*

$$
\mathrm{P}^{\mathrm{SB}} = I\sum_{i=1}^{K-1} P(\bar{t}_i)Q(\tau^m(x) > \bar{t}_i)(B^1_i - B^2_i),
$$

with $B^1_1 = 1$ *and*

$$
B^1_i = \frac{e^{-C(\bar{t}_i-\bar{t}_1)}}{(2\pi)^{i-1}}\int_{\mathbb{R}^{i-1}} M^i(u,T)du,
$$

for $i \in \{2,\dots,K-1\}$, *and*

$$
B^2_i = \frac{e^{-C(\bar{t}_{i+1}-\bar{t}_1)}}{(2\pi)^i}\int_{\mathbb{R}^i} N^i(u,T)du
$$

for $i \in \{1,\dots,K-1\}$.

Proof. Given the payoff function in equation (10), the relevant expression for all surrender time points $\bar{t}_i$ is

$$
E_Q\Big[e^{-\int_0^{\bar{t}_i} r(s)ds}\mathbb{1}_{\{\tau^s=\bar{t}_i\}}\mathbb{1}_{\{\tau^s<\tau^m(x)\}}S_{\bar{t}_i})\Big] = Q(\tau^m(x) > \bar{t}_i)\, E_Q\Big[e^{-\int_0^{\bar{t}_i} r(s)ds}\mathbb{1}_{\{\tau^s=\bar{t}_i\}}S_{\bar{t}_i}\Big]. \quad (33)
$$

Observe that $\mathbb{1}_{\{\tau^s=\bar{t}_i\}} = \mathbb{1}_{\{\bar{t}_i\le\tau^s\}} - \mathbb{1}_{\{\bar{t}_{i+1}\le\tau^s\}}$. Consequently in virtue of equation (16), (33) can be written as

$$
\begin{aligned}
E_Q\Big[e^{-\int_0^{\bar{t}_i} r(s)ds}\mathbb{1}_{\{\tau^s=\bar{t}_i\}}\mathbb{1}_{\{\tau^s<\tau^m(x)\}}S_{\bar{t}_i}\Big] &= Q(\tau^m(x) > \bar{t}_i)\, E_Q\Big[e^{-\int_0^{\bar{t}_i} r(u)du}S_{\bar{t}_i}e^{-\int_0^{\bar{t}_i}\lambda^s(u)du}\Big]\\
&\quad - Q(\tau^m(x) > \bar{t}_i)\, E_Q\Big[e^{-\int_0^{\bar{t}_i} r(u)du}S_{\bar{t}_i}e^{-\int_0^{\bar{t}_{i+1}}\lambda^s(u)du}\Big].
\end{aligned}
$$

For short, we denote the first expectation by B^1_i and the second one by B^2_i. Let $Q^{S,i}$, $i = 1,\dots,K-1$ be the spot probability measure defined by

$$
\frac{dQ^{S,i}}{dQ} = e^{-\int_0^{\bar{t}_i} r(u)du}S_{\bar{t}_i}. \quad (34)
$$

Note that this defines indeed a density process as the discounted stock price $e^{-\int_0^t r(u)du}S_t$ is a Q-martingale. Using this new measure B_i^1 and B_i^2 simplify to

$$
\begin{aligned}
B_i^1 &= E_{Q^{S,i}}\left[e^{-\int_0^{\bar{t}_i}\lambda^s(u)du}\right], && (35)\\
B_i^2 &= E_{Q^{S,i}}\left[e^{-\int_0^{\bar{t}_{i+1}}\lambda^s(u)du}\right]. && (36)
\end{aligned}
$$

The above expressions are well-defined as $\lambda^s(u)$ for $\bar{t}_i \le u < \bar{t}_{i+1}$ is defined by $D(\bar{t}_i)$ and consequently is $\mathscr{F}_{\bar{t}_i}^{L^1,L^2}$-measurable. The calculation of expectation (35) follows the same argument used for the calculation of A_1 in the proof of Theorem 4.1. $B_1^1 = 1$ as by construction $\lambda^s(u) = 0$ for $u \in [0, \bar{t}_1)$. For $i \in \{2, \dots, K-1\}$, we get

$$
e^{C(\bar{t}_i-\bar{t}_1)}E_{Q^{S,i}}\left[e^{-\int_0^{\bar{t}_i}\lambda^s(u)du}\right] = E_{Q^{S,i}}\left[f^{i-1}(D(\bar{t}_1), \dots, D(\bar{t}_{i-1}))\right], \tag{37}
$$

with $f^{i-1}(x_1, \dots, x_{i-1}) := \prod_{l=1}^{i-1} f_l(x_l)$. The latter expectation is

$$
\frac{1}{(2\pi)^{i-1}}\int_{\mathbb{R}^{i-1}} \tilde{M}^{i-1}(\mathrm{i}u)\widehat{f^{i-1}}(-u)du, \tag{38}
$$

for

$$
\widehat{f^{i-1}}(u_1, \dots, u_{i-1}) = \prod_{l=1}^{i-1}\hat{f}_l(u_l),
$$

and $\tilde{M}^{i-1}(\mathrm{i}u)$ defined as

$$
\tilde{M}^{i-1}(\mathrm{i}u) = E_{Q^{S,i}}\left[e^{\mathrm{i}u_1D(\bar{t}_1)+\ldots+\mathrm{i}u_{i-1}D(\bar{t}_{i-1})}\right].
$$

It follows from (21) that

$$
\begin{aligned}
\tilde{M}^{i-1}(\mathrm{i}u) &= E_Q\left[e^{\mathrm{i}u_1D(\bar{t}_1)+\ldots+\mathrm{i}u_{i-1}D(\bar{t}_{i-1})+\int_0^{\bar{t}_i}\sigma_2(s)dL_s^2+\int_0^{\bar{t}_i}\beta(s)dL_s^1-\omega(\bar{t}_i)}\right]\\
&= \exp\Big(\mathrm{i}\sum_{l=1}^{i-1}u_l(-p(\bar{t}_l)-\delta T+\int_0^T f(0,s)ds+\int_0^{\bar{t}_l}A(s,T)ds-\omega(\bar{t}_l))\Big)\\
&\quad\times E_Q\Big[\exp\Big(\mathrm{i}\sum_{l=1}^{i-1}\Big(\int_0^{\bar{t}_l}u_l\sigma_2(s)dL_s^2+\int_0^{\bar{t}_l}u_l(\beta(s)-\Sigma(s,T))dL_s^1\Big)\\
&\qquad+\int_0^{\bar{t}_i}\sigma_2(s)dL_s^2+\int_0^{\bar{t}_i}\beta(s)dL_s^1-\omega(\bar{t}_i)\Big)\Big].
\end{aligned}
$$

Using (B.6) and (A.2), the last expectation becomes

$$
\begin{aligned}
&E_Q\Big[\exp\Big(\int_0^{\bar{t}_i}E^i(s,u,T)dL_s^1+\int_0^{\bar{t}_i}F^i(s,u)dL_s^2\Big)\Big]\\
&\qquad= \exp\Big(\int_0^{\bar{t}_i}\Big(\theta_s^1(E^i(s,u,T))+\theta_s^2(F^i(s,u))\Big)ds\Big)
\end{aligned}
$$

The representation of B_i^1 in the Theorem follows from (38) as

$$
\tilde{M}^{i-1}(\mathrm{i}u) = D^i(u,T)\exp\Big(\int_0^{\bar{t}_i}\Big(\theta_s^1(E^i(s,u,T))+\theta_s^2(F^i(s,u))\Big)ds\Big).
$$

The expression for B_i^2 is obtained by the same argument as

$$B_i^2 \;=\; e^{-C(\bar{t}_{i+1}-\bar{t}_1)} E_{Q^{S,i}}\Big[f^i(D(\bar{t}_1),\ldots,D(\bar{t}_i))\Big]. \tag{39}$$

□

Using the same mathematical tools, we can derive the term structure of the probability of surrender. For this purpose, we consider the complement probability

$$Q\left(\tau^s \geq \bar{t}_i\right) = \mathbb{E}_Q\left[\exp\left(-\int_0^{\bar{t}_i} \lambda^s(u)du\right)\right],$$

which follows from equation (16).

Proposition 4.4. *The probability of no surrender before $\bar{t}_i$ is given by*

$$Q\left(\tau^s \geq \bar{t}_i\right) = \frac{e^{-C(\bar{t}_i-\bar{t}_1)}}{(2\pi)^{i-1}} \int_{\mathbb{R}^{i-1}} \bar{M}^i(u,T)du, \tag{40}$$

with

$$\bar{M}^i(u,T) = D^{i-1,i}(u,T)\exp\left(\int_0^{\bar{t}_i}\left(\theta_s^1\left(E^i(s,u,T)-\beta(s)\right)+\theta_s^2\left(F_{i-1}(s,u)\right)\right)ds\right)\widehat{f^{i-1}}(-u),$$

$D^{i-1,i}(u,T)$ and $F_{i-1}(s,u)$ defined in (B.5), and $E^i(s,u,T)$ is given in (B.6).

Proof. Exploiting the structure of the surrender intensity λ^s, we obtain

$$Q\left(\tau^s \geq \bar{t}_i\right) = e^{-C(\bar{t}_i-\bar{t}_1)} E_Q\Big[f^{i-1}(D(\bar{t}_1),\ldots,D(\bar{t}_{i-1}))\Big]$$

with $f^{i-1}(x_1,\ldots,x_{i-1}) := \prod_{l=1}^{i-1} f_l(x_l)$. The proof follows now the same lines as for the derivation of B_i^1 in the previous proof. The only difference being we now operate under the risk neutral measure Q instead of the spot measure $Q^{S,i}$. □

5. Numerical analysis and Results

5.1. **Setup.** In this section we discuss the numerical implementation of the results obtained in section 4. We stress that Theorems 4.1 – 4.3 hold for a large variety of driving processes within the class of Lévy processes, such as hyperbolic, Normal Inverse Gaussian, Variance Gamma and CGMY; for illustration purposes the following assumptions are in place.

The financial market. We choose as a relevant Lévy process the Normal Inverse Gaussian (NIG) process with cumulant function

$$\theta(u) = \mu u + \delta'\left(\sqrt{(\alpha')^2-(\beta')^2}-\sqrt{(\alpha')^2-(\beta'+u)^2}\right), \quad -\alpha'-\beta' < u < \alpha'-\beta',$$

for $\mu \in \mathbb{R}$, $\delta' > 0$, $0 \leq |\beta'| < \alpha'$. The parameter α' controls the steepness of the density (and therefore its tail behaviour), β' controls the skewness of the distribution, whilst δ' is the scale parameter; the location parameter μ is instead set to zero, without loss of generality.

Further, we assume a simplified Vašiček structure for the function $\sigma_1(s,T)$ so that for $a > 0$

$$\sigma_1(s,T) = \begin{cases} ae^{-a(T-s)}, & s \leq T \\ 0, & s > T \end{cases} \qquad \Sigma(s,T) = \begin{cases} 1-e^{-a(T-s)}, & s \leq T \\ 0, & s > T \end{cases}.$$

For the equity part, we assume $\sigma_2(s) = \sigma_2 > 0$ and $\beta(s) = b \in \mathbb{R}$.

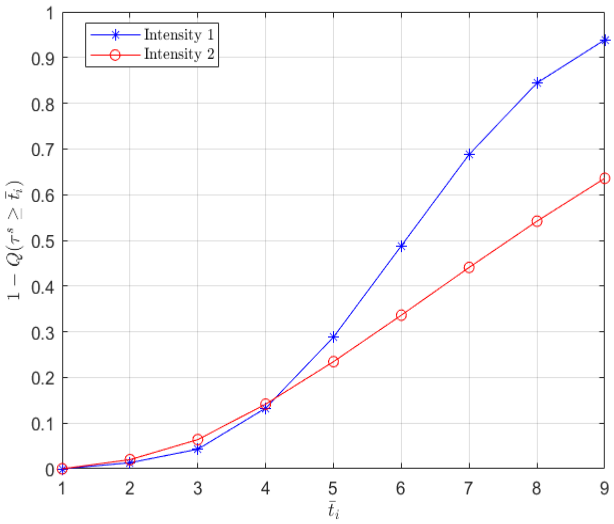

FIGURE 1. Term structure of surrender probabilities. ‘Intensity 1’: $h(x) = x^2$. ‘Intensity 2’: $h(x) = |x|$. Parameters: Table 1.

The surrender intensity function. Although the setting is fairly general concerning the function of the spread $D(t)$ in the construction of the surrender intensity, for illustration purposes we focus on the cases $h(x) = x^2$ and $h(x) = |x|$. The resulting versions are labelled ‘Intensity 1’ and ‘Intensity 2’ respectively.

It follows in the first case (‘Intensity 1’) that

$$f_l(x) = e^{-\beta\Delta\bar{t}_{l+1}h(x)} = e^{-\beta\Delta\bar{t}_{l+1}x^2},$$

which implies, by recognizing a rescaled Gaussian distribution, that

$$\hat{f}_l(y) = \sqrt{\frac{\pi}{\beta\Delta\bar{t}_l}}\exp\Big(-y^2/(4\beta\Delta\bar{t}_l)\Big), \quad y \in \mathbb{C}. \tag{41}$$

In the second case (‘Intensity 2’) instead,

$$f_l(x) = e^{-\beta\Delta\bar{t}_{l+1}h(x)} = e^{-\beta\Delta\bar{t}_{l+1}|x|};$$

a straightforward integration shows that

$$\hat{f}_l(y) = \frac{2\beta\Delta\bar{t}_{l+1}}{y^2 + (\beta\Delta\bar{t}_{l+1})^2}, \quad y \in \mathbb{C}. \tag{42}$$

Note that this result could also be recovered in virtue of the properties of the Cauchy distribution.

We note that in both specifications the function $h(x)$ is non-negative and U-shaped, but characterised by a different rate of growth. This is reflected in the resulting term structure of surrender probabilities shown in Figure 1. This will allow us to assess the impact on the value of the VA and its components of expert judgement concerning the choice of the shape of the surrender intensity.

Parameters. All numerical experiments refer to contracts with a parameter setting as in Table 1. Various values of the participation rate, ξ, and the cap rate, γ, are considered in the following analysis.

TABLE 1. Parameters for the Variable Annuity contract. Reference process: NIG. Financial Market Model parameters - source: Eberlein and Rudmann (2018). Mortality Model parameters - source: Escobar et al. (2016).

Variable Annuity		Financial Market Model			Surrender Model		Mortality Model	
			L_t^1	L_t^2				
I	100	α'	4	5.73	β	0.05	q	12.1104
T	10 years	β'	-3.8	-2.13	C	0.01	z	76.139
φ	0.0105	δ'	1.34	8.3			κ	0.4806
$P(\bar{t}_l)$	$0.95 + 0.05\bar{t}_l/T$	a	0.0020898	-			λ	0.0195
$\Delta t_j = \Delta \bar{t}_l$	1 year	σ_2	-	0.1818			σ	0.0254
$\tilde{t}_i - \tilde{t}_{i-1}$	6 months	b	-	0.0065				

The parameters of the financial model are taken from the calibration exercise in Eberlein and Rudmann (2018). Further, for illustration purposes, we consider the case of a VA policy with annual ratchet reset schedule; surrender and mortality are instead monitored annually and semi-annually respectively. We observe though that the results presented in section 4 hold for any other choice of the relevant time grids.

All experiments are run in parallel on the Camber cluster at City, University of London.

5.2. **Implementation.** Theorems 4.1 – 4.3 offer analytical pricing formulas in terms of multidimensional integrals; the dimensionality depends on the maturity of the policy and the frequency with which the surrender event is monitored. For example, for the case under consideration of a 10 year contract with annual surrender monitoring, the pricing formulas are expressed in terms of integrals with up to 9 dimensions. VA with longer maturities - which is usually the case - and/or higher monitoring frequency will be characterised by integrals with more dimensions. As deterministic quadrature methods, such as global adaptive quadrature, can be implemented in practice for integral with at most 3 dimensions (see Brandimarte, 2013, for example), we adopt Monte Carlo integration with importance sampling for the computation of the valuation formulas in Theorems 4.1 – 4.3. Monte Carlo integration is based on the construction of random grids in the hyperspace, paired with interpreting the integral as an expected value (see Brandimarte, 2013, for further details). Thus, the multidimensional integral over the domain $\Omega \subset \mathbb{R}^d$

$$I = \int_{\Omega} f(x)dx$$

is approximated by

$$I_{MC} = \frac{1}{M}\sum_{i=1}^{M}\frac{f(x_i)}{p(x_i)},$$

i.e. by evaluating the function $f(x)$ at M points, x, drawn randomly in Ω with a probability density $p(x)$. The approximation error is measured by the (unbiased) sample variance

$$\frac{\sum_{i=1}^{M}\left(I_i - I_{MC}\right)^2}{M-1}.$$

Consequently, the rate of convergence crucially depends on the sample size M instead of the dimension of the integral.

In order to speed up convergence, we adopt importance sampling as the most adequate variance reduction technique to tackle oscillatory integrand functions, such as the ones in Theorems 4.1 – 4.3. In Figure 2 we illustrate the case of the integrand functions defining

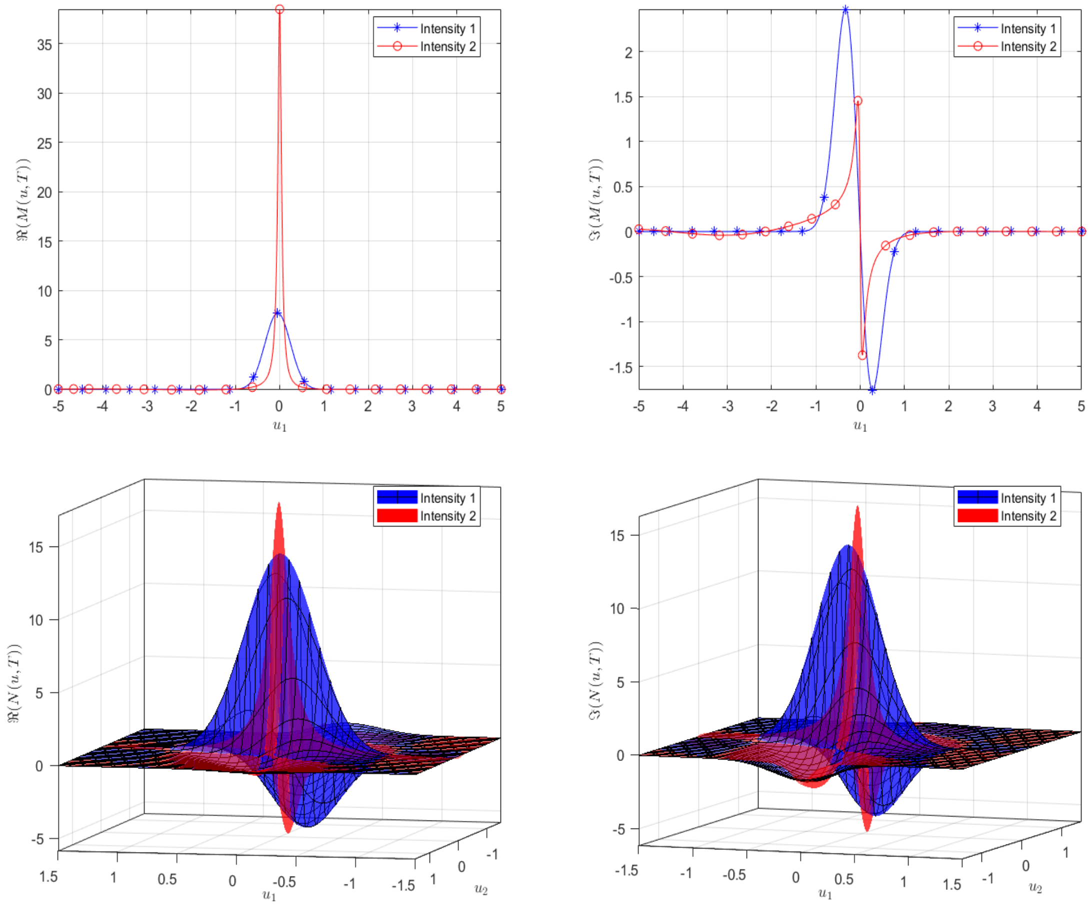

FIGURE 2. $M(u,T)$ and $N(u,j,\varphi/\xi,T)$ - Theorem 4.1. Intensity 1: $h(x)=x^2$. Intensity 2: $h(x)=|x|$. Maturity $T=3$, $j=1$, $\xi=0.7$; other parameters: Table 1. Top panels: real and imaginary part of $M(u,T)$. Bottom panels: real and imaginary part of $N(u,j,\varphi/\xi,T)$.

a 3 years GMAB for both surrender intensity versions considered in this section. We note that, although all functions are centered around the origin, the $M(\cdot)$ and $N(\cdot)$ functions generated under 'Intensity 2' are more strongly peaked and faster decaying. These different shapes and oscillatory behaviours are primarily due to the different Fourier transforms $\hat{f}$ involved (see eq. (41) and (42)), which can be linked to a rescaled Gaussian and Cauchy distribution, respectively. Consequently, the choice of the importance sampling distribution changes according to the adopted surrender intensity.

In details, for $h(x)=x^2$, i.e. for 'Intensity 1', we choose the multivariate Gaussian distribution with zero mean, independent components, and a given variance matrix, which is treated as a parameter. Thus, for $d>0$ indicating the required dimension,

$$p(x)=\prod_{j=1}^{d}\frac{1}{(2\pi s_j^2)^{1/2}}\,e^{-\frac{x_j^2}{2s_j^2}}.$$

Numerical experiments show that relatively small biases and standard errors for GMAB and DB can be obtained by using the same variance for the first $K-1$ dimensions fixed at $s_j^2=(0.5)^2$, $j=1,\dots,K-1$, and increasing this value to $s_K^2=(2.1)^2$ for the K^{th} dimension, as to cater for the oscillatory behaviour induced by the Fourier transform of the option payoff. As far as the SB is concerned, the variance is fixed at 0.16 across all dimensions.

For the case of $h(x) = |x|$, i.e. for 'Intensity 2', we adopt instead the multivariate Cauchy distribution with independent components, zero location parameter, and a scale parameter set at 0.05 for all dimensions, so that

$$p(x) = \prod_{j=1}^{d} \frac{s_j}{\pi\left(x_j^2 + s_j^2\right)}, \quad s_j = 0.05,\ j = 1, \ldots, d.$$

Numerical experiments show that this choice minimizes the Monte Carlo standard error.

In order to test the accuracy of the Monte Carlo integration procedure introduced above, we first use our proposed algorithms to price a short term contract, for which integration via standard quadrature packages is possible. Therefore, we consider the case of a 3 year GMAB; as $K = 2$, and the surrender dates $\bar{t}_l$ are annually spaced, the expressions for A_1 and $A_{2,\cdot}$ in Theorem 4.1 are given by one- and two-dimensional integrals respectively, which can be computed by any standard software package.

Results for the GMAB are reported in the first five columns of Table 2 for both intensity functions considered in this paper (panel A and B respectively). The Table reports values obtained with Monte Carlo integration, as well as the corresponding values from deterministic quadrature methods. We also report measures of the accuracy of the Monte Carlo integral, I_{MC}, both in terms of the absolute value of the bias expressed as percentage of the value obtained by quadrature I_Q, i.e.

$$100 \times \frac{|I_{MC} - I_Q|}{I_Q},$$

and the percentage standard error

$$100 \times \frac{1}{I_{MC}} \sqrt{\frac{\sum_{i=1}^{M}(I_i - I_{MC})^2}{M(M-1)}}.$$

The numerical results confirm the quality of the procedure as all biases and standard errors are below 0.5%, and this also applies to the value of the full contract. Similar results are obtained for both the DB and SB (see Table E.1 in (online) Appendix E).

We then consider a 10-year contract (for which deterministic quadrature methods are no longer of practical use); results are presented in Table 3. At this stage, we note from the first five columns the low standard errors characterizing the values of all the components of the VA regardless of the model chosen for the surrender intensity. However, such a low error is achieved at the cost of a high CPU time, especially in the case of the DB component. This is primarily due to the computation of the quantities $\int_0^T \theta_s^1(E(s,u,T))ds$, $\int_0^T \theta_s^1(\tilde{E}(s, v - \mathrm{i}R, j, T))ds$ in Theorems 4.1 and the analogous integrals in Theorems 4.2 and 4.3.

For the purpose of speeding up the computations, we tabulate these integrals over a grid of values once for use in all simulations. Specifically, for the first integral we consider a grid of values $z = \sum_{l=1}^{K-1} u_l 1_{\{0 \le s \le \bar{t}_l\}}$. For the second integral, we consider a 2-D grid of values (z_1, z_2), with $z_1 = \sum_{l=1}^{K-1} v_l 1_{\{0 \le s \le \bar{t}_l\}}, z_2 = v_K$. The integrals in Theorems 4.2 and 4.3 are dealt with in a similar manner. Then, we use spline interpolation during the Monte Carlo iterations, with linear extrapolation. We label this routine MCi for convenience.

In order to assess the quality of this new approximation, we recalculate the values of the components of GMAB, DB and SB for the short maturities examples. The results are

Table 2. Benchmarking Monte Carlo integration with importance sampling: GMAB, maturity $T = 3$ years. Cap rate: $\gamma = 0.05$, participation rate $\xi = 0.7$. Other parameters: Table 1. 'Quadrature': Matlab built-in functions `integral`, and `integral2`. 'MC': Monte Carlo integration. 'MCi': Monte Carlo integration with interpolation. Bias/standard error expressed as percentage of the actual value. Monte Carlo iterations: 100 batches of size 10^6.

GMAB

A. Intensity 1

	Quadrature	MC	bias %	std. err. %	MCi	bias %	std. err. %
A_1	0.9867	0.9867	0.0015	0.0050	0.9867	0.0008	0.0050
$A_{2,1}(\varphi/\xi)$	0.0887	0.0887	0.0050	0.1494	0.0888	0.1016	0.1447
$A_{2,2}(\varphi/\xi)$	0.0889	0.0889	0.0095	0.1489	0.0890	0.1056	0.1443
$A_{2,3}(\varphi/\xi)$	0.0893	0.0892	0.0102	0.1484	0.0894	0.1040	0.1437
$A_{2,1}(\gamma/\xi)$	0.0667	0.0667	0.0394	0.1986	0.0668	0.1994	0.1932
$A_{2,2}(\gamma/\xi)$	0.0669	0.0669	0.0324	0.1978	0.0670	0.2010	0.1924
$A_{2,3}(\gamma/\xi)$	0.0672	0.0672	0.0315	0.1970	0.0673	0.1992	0.1916
GMAB	102.7713	102.7666	0.0046	0.0512	102.7669	0.0044	0.0515

B. Intensity 2

	Quadrature	MC	bias %	std. err. %	MCi	bias %	std. err. %
A_1	0.9799	0.9799	0.0006	0.0008	0.9799	0.0001	0.0008
$A_{2,1}(\varphi/\xi)$	0.0880	0.0879	0.1031	0.1282	0.0883	0.3597	0.1238
$A_{2,2}(\varphi/\xi)$	0.0883	0.0882	0.1027	0.1275	0.0882	0.0598	0.1235
$A_{2,3}(\varphi/\xi)$	0.0886	0.0886	0.1022	0.1269	0.0886	0.0595	0.1229
$A_{2,1}(\gamma/\xi)$	0.0661	0.0660	0.1443	0.1807	0.0664	0.3964	0.1746
$A_{2,2}(\gamma/\xi)$	0.0664	0.0663	0.1440	0.1794	0.0664	0.0834	0.1740
$A_{2,3}(\gamma/\xi)$	0.0667	0.0666	0.1433	0.1785	0.0667	0.0829	0.1731
GMAB	102.0682	102.0686	0.0004	0.0147	102.0721	0.0039	0.0147

shown in the final three columns of Tables 2, and confirm that MCi offers a comparable degree of accuracy. For the case of the 10 year VA, Table 3 (final four columns) highlights the significant computational advantage of the interpolation procedure, which is achieved maintaining the standard errors in the same range of magnitude.

5.3. **Analysis.** Given the complex design of the ratchet variable annuity, we employ the MCi scheme to analyse some of the contract features and the consequent management implications. Thus, we focus in particular on the impact of the alternative constructions of the surrender intensity, and the role of the participation rate and the cap rate.

From Table 3, we observe that the SB is more expensive by more than 25% under 'Intensity 2' compared to 'Intensity 1'. These results offer an insight into the impact of the surrender model risk, i.e. the risk of losses resulting from using inadequate modelling assumptions. As 'Intensity 2' attributes more weight to the risk of early termination, the corresponding values of the GMAB and DB are lower compared to 'Intensity 1', although this difference is smaller and ranges between 3% to 5%.

In Table 4 we study the impact of varying cap and participation rates. As in our construction the payoff of the surrender benefit does not depend on these parameters, we focus on

TABLE 3. Variable Annuity contract and its components: $T = 10$ years. Value, standard errors and CPU time (expressed in seconds and referred to the average time of 1 batch of 10^6 iterations across 100 batches of the same size). MC: Monte Carlo integration. MCi: Monte Carlo integration with interpolation. Parameters: Table 1; cap rate: $\gamma = 0.05$, participation rate $\xi = 0.7$.

A. Intensity 1

	MC				MCi			
	GMAB	DB	SB	VA	GMAB	DB	SB	VA
Value	83.1489	12.8546	15.4517	111.4551	83.1423	12.8547	15.4432	111.4402
Std. Err. (%)	0.2518	0.0458	0.0521	0.1881	0.2577	0.0435	0.0521	0.1925
CPU (sec.)	1493.30	6463.70	358.53	8315.53	125.66	706.30	36.21	868.18

B. Intensity 2

	MC				MCi			
	GMAB	DB	SB	VA	GMAB	DB	SB	VA
Value	78.9747	12.4468	19.4509	110.8725	78.9745	12.4483	19.4515	110.8743
Std. Err. (%)	0.0512	0.0078	0.0132	0.0366	0.0515	0.0079	0.0132	0.0367
CPU (sec.)	917.36	6160.61	229.42	7307.39	76.02	548.76	24.74	649.52

TABLE 4. Cap rate vs participation rate: GMAB and DB. Monte Carlo integration with interpolation. Parameters: Table 1.

A. Intensity 1

		GMAB					DB				
	ξ	0.3	0.5	0.7	0.9	1	0.3	0.5	0.7	0.9	1
	0.05	81.0250	82.4851	83.1423	83.5408	83.6763	12.6637	12.7944	12.8547	12.8892	12.9015
γ	0.075	84.3277	87.3370	88.8007	89.6653	89.9523	12.9467	13.2147	13.3454	13.4219	13.4492
	0.2	89.2550	99.6493	106.6142	111.2983	113.0813	13.3685	14.2652	14.8754	15.2904	15.4493

B. Intensity 2

		GMAB					DB				
	ξ	0.3	0.5	0.7	0.9	1	0.3	0.5	0.7	0.9	1
	0.05	76.9293	78.3300	78.9745	79.3427	79.4731	12.2654	12.3905	12.4481	12.4811	12.4928
γ	0.075	80.0400	82.9219	84.3230	85.1552	85.4326	12.5376	12.7942	12.9193	12.9924	13.0185
	0.2	84.7203	94.5085	101.1225	105.5918	107.3008	12.9444	13.8055	14.3918	14.7881	14.9401

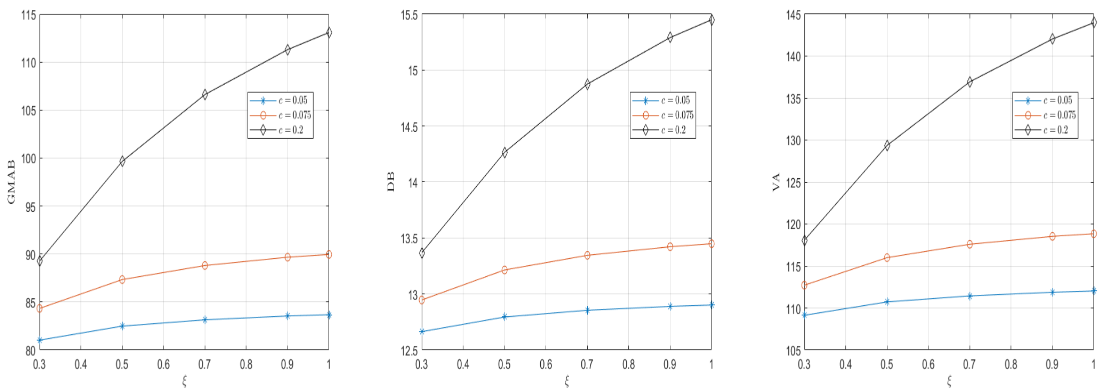

FIGURE 3. Sensitivity Analysis: participation rate ξ, cap rate γ. Left-hand-side panel: GMAB. Mid-panel: DB. Right-hand-side panel: VA. Intensity 1. Parameters: Table 1.

the values of the GMAB and the DB components. The case 'Intensity 1' is also illustrated in Figure 3, in which we show the corresponding value of the full VA policy as well.

The participation rate ξ represents a key parameter for the marketability of the annuity contract; however, the results in Table 4 show that the reward to the policyholder of the participation in the growth of the economy is significantly affected by the cap rate γ written by the insurance company. Indeed, in presence of a tight cap rate, the value of the benefits are almost unaffected by ξ: for $\gamma = 0.05$ the change in value of the GMAB between the case of a policyholder offered a 30% participation rate and one offered a 100% participation is 3.3%. Relaxation of the cap rate, on the other hand, can significantly enhance the attractiveness of the benefits: moving the cap rate from 0.05 to 0.075 results already in a 6.7% difference between the two policies. This difference is even more significant if we apply a cap $\gamma = 0.20$, which could be considered almost as a 'no-cap' case, as it rises to 26.7%. The insurance company on the other hand would be more exposed to the market risk: the ratchet feature locks in the returns at the end of every year, and adequate reserves need to be set aside to meet the liability at expiration. This can prove quite challenging in fast changing market conditions. Similar considerations hold for the DB part of the policy.

These results show that the cap rate γ is an important 'knob' for the insurer to fine tune the level on the tradeoff between the risk exposure generated by these products, and the return and effective marketability of the products themselves.

6. Conclusions

We proposed a framework for the valuation and management of complex life insurance contracts, which we illustrate by means of a detailed study of ratchet variable annuities. We investigate the tradeoff between the insurer's risk exposure and the marketability of these products. Our analysis highlights in particular the role of cap rates.

Variable insurance products provide an income stream during retirement and other benefits for policyholders; however, they are usually characterized by complex designs, which can be associated with high risks. Hence, the recent decision of the Securities and Exchange Commission (SEC) to offer a prospectus which could help investors to better understand such sophisticated policies (SEC, 2020). The proposed methodology could represent a valid support for the development of such overviews, as it would give the opportunity to gain insights into the role of certain key parameters, and the impact of expert knowledge in the modelling of the surrender behaviour.

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Appendix A. Time-inhomogenous Lévy processes.

Let $(\Omega, \mathscr{F}, \mathbb{F}, Q)$ be a filtered probability space, with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,T^*]}$, for a given finite time horizon $T^* > 0$, satisfying the usual conditions. The probability measure Q is interpreted as a risk-neutral martingale measure. We consider a time-inhomogeneous Lévy process, L, with characteristic function

$$E_Q\Big[e^{\mathrm{i}uL_t}\Big] = \exp\bigg(\int_0^t \Big(\mathrm{i}ub_s - \frac{1}{2}c_s u^2 + \int_{\mathbb{R}} \big(e^{\mathrm{i}ux} - 1 - \mathrm{i}ux\big)F_s(dx)\Big)ds\bigg). \tag{A.1}$$

Assume that the local characteristics $(b_s, c_s, F_s)_{s\in[0,T^*]}$ satisfy the integrability condition

$$\int_0^{T^*} \Big(|b_s| + c_s + \int_{\mathbb{R}} (\min\{|x|^2, 1\})F_s(dx)\Big)ds < \infty.$$

Note that we do not need any truncation function h such as $\mathbb{1}_{\{|x|\le 1\}}$ in (A.1) as we always assume that the exponential moments of a certain order exist in the following sense.

Assumption A.1 (Exponential moments)**.** There exist positive constants M and ϵ such that for each $u \in [-(1+\epsilon)M, (1+\epsilon)M]$

$$\int_0^{T^*} \int_{\{|x|>1\}} e^{ux}\, F_s(dx)ds < \infty.$$

The above assumption is satisfied by all standard processes typically used in mathematical finance such as hyperbolic, Normal Inverse Gaussian, Variance Gamma and CGMY processes.

We further consider the cumulant function of L,

$$\theta_s(z) = b_s z + \frac{1}{2}c_s z^2 + \int_{\mathbb{R}} (e^{zx} - 1 - zx)F_s(dx),$$

which is defined for any $z \in \mathbb{C}$ such that $Re(z) \in [-(1+\epsilon)M, (1+\epsilon)M]$. For those $z \in \mathbb{C}$, $E_Q[\exp(zL_t)] < \infty$ and

$$E_Q\big[\exp(zL_t)\big] = \exp\bigg(\int_0^t \theta_s(z)ds\bigg).$$

Finally, for any left-continuous function $f : \mathbb{R}^+ \to \mathbb{C}$ with $|Re(f)| \le M$, the following holds

$$E_Q\bigg[\exp\bigg(\int_0^t f(s)dL_s\bigg)\bigg] = \exp\bigg(\int_0^t \theta_s(f(s))ds\bigg), \tag{A.2}$$

where the integrals are defined component-wise for the real and imaginary part. For the derivation of this formula, see Eberlein and Raible (1999). As the Riemann sums which are used to approximate the stochastic integral are defined on the basis of predictable step functions, the proof still holds for left-continuous functions f. A more general version of this formula is given by Proposition 4.8 in Eberlein and Kallsen (2019); the proof exploits the explicit form of the local characteristics of the stochastic integral. See also Example 4.18 in Eberlein and Kallsen (2019) for further details.

Appendix B. Auxiliary functions for the Theorems in Section 4

B.1. **Functions for Theorem 4.1.** For $i = 1, \ldots, K$, let us define the function f^i on $\mathbb{R}^i$

$$f^i(x_1, \ldots, x_i) = \prod_{l=1}^{i} f_l(x_l). \tag{B.1}$$

Let

$$w_l := \int_0^{\bar{t}_l} A(s,T)ds + \int_0^T f(0,s)ds - \delta T - \omega(\bar{t}_l) - p(\bar{t}_l) \tag{B.2}$$

for $l = 1, \ldots, K-1$ and

$$\widetilde{w}_{t_j} := \int_{]t_{j-1},t_j]} f(0,s)ds + \int_0^{t_j} A(s,t_j)ds - \int_0^{t_{j-1}} A(s,t_{j-1})ds - (\omega(t_j) - \omega(t_{j-1})), \tag{B.3}$$

for $j = 1, \ldots, N$.

Further, define $R := (0, \ldots, 0, r) \in \mathbb{R}^K$, with $1 < r < 2$, and let for all $0 \le s \le T$, and $u \in \mathbb{R}^{K-1}$, $v \in \mathbb{C}^K$, $j = 1, \ldots, N$

$$\begin{aligned} D(u,T) &:= \exp\Big(\mathrm{i}\sum_{l=1}^{K-1} u_l w_l\Big), \\ \tilde{D}(v,j,T) &:= D(v_1, \ldots, v_{K-1}, T)\exp\Big(\mathrm{i}v_K \widetilde{w}_{t_j}\Big), \end{aligned}$$

$$\begin{aligned} E(s,u,T) &:= \Sigma(s,T) + \mathrm{i}(\beta(s) - \Sigma(s,T))\sum_{l=1}^{K-1} u_l \mathbb{1}_{\{0\le s\le \bar{t}_l\}}, \\ \tilde{E}(s,v,j,T) &:= E(s,v_1,\ldots,v_{K-1},T) + \mathrm{i}\big(\beta(s)\mathbb{1}_{\{t_{j-1}<s\le t_j\}} + \Sigma(s,t_{j-1})\mathbb{1}_{\{0\le s\le t_{j-1}\}} \\ &\qquad - \Sigma(s,t_j)\mathbb{1}_{\{0\le s\le t_j\}}\big)v_K, \end{aligned}$$

$$\begin{aligned} F(s,u) &:= \mathrm{i}\sigma_2(s)\sum_{l=1}^{K-1} u_l \mathbb{1}_{\{0\le s\le \bar{t}_l\}}, \\ \tilde{F}(s,v,j) &:= F(s,v_1,\ldots,v_{K-1}) + \mathrm{i}\sigma_2(s)\mathbb{1}_{\{t_{j-1}<s\le t_j\}} v_K, \end{aligned} \tag{B.4}$$

$$M(u,T) := D(u,T)\ \exp\left(\int_0^T \left(\theta_s^1(E(s,u,T)) + \theta_s^2(F(s,u))\right) ds\right)\ \widehat{f^{K-1}}(-u),$$

$$\begin{aligned} N(v,j,\kappa,T) &:= \tilde{D}(v-\mathrm{i}R,j,T)\ \exp\left(\int_0^T \Big(\theta_s^1(\tilde{E}(s,v-\mathrm{i}R,j,T)) + \theta_s^2(\tilde{F}(s,v-\mathrm{i}R,j))\Big) ds\right) \\ &\quad \times \widehat{f^{K-1}}(-v_1,\ldots,-v_{K-1})\frac{(1+\kappa)^{1-\mathrm{i}v_K-r}}{(\mathrm{i}v_K+r-1)(\mathrm{i}v_K+r)}, \text{ for } v \in \mathbb{R}^K. \end{aligned}$$

B.2. **Functions for Theorem 4.2.** For $j \in \{1, \ldots, K-1\}$, let $R := (0, \ldots, 0, r) \in \mathbb{R}^{j+1}$ with $1 < r < 2$, $u \in \mathbb{R}^j$, and $v \in \mathbb{C}^{j+1}$. For $0 \le s \le T$, $j \in \{1, \ldots, K-2\}$ and all $i \in \{1, \ldots, N\}$ such that $\bar{t}_j < \tilde{t}_i \le \bar{t}_{j+1}$, as well as for $j = K-1$ and all i such that $\bar{t}_{K-1} < \tilde{t}_i \le T$, let

$$\begin{aligned} D^{j,i}(u,T) &:= \exp\Big(\mathrm{i}\sum_{l=1}^{j} u_l w_l\Big), \\ \tilde{D}^{j,i}(v,l',T) &:= D^{j,i}(v_1,\ldots,v_j,T)\exp\Big(\mathrm{i}v_{j+1}\widetilde{w}_{t_{l'}}\Big), \end{aligned}$$

$$\begin{aligned} E_{j,i}(s,u,T) &:= \Sigma(s,\tilde{t}_i) + \mathrm{i}(\beta(s) - \Sigma(s,T))\sum_{l=1}^{j} u_l \mathbb{1}_{\{0\le s\le \bar{t}_l\}}, \\ \tilde{E}_{j,i}(s,v,l',T) &:= E_{j,i}(s,v_1,\ldots,v_j,T) \\ &\quad + \mathrm{i}\big(\beta(s)\mathbb{1}_{\{t_{l'-1}<s\le t_{l'}\}} + \Sigma(s,t_{l'-1})\mathbb{1}_{\{0\le s\le t_{l'-1}\}} - \Sigma(s,t_{l'})\mathbb{1}_{\{0\le s\le t_{l'}\}}\big)v_{j+1}, \end{aligned}$$

$$\begin{aligned} F_j(s,u) &:= \mathrm{i}\sigma_2(s)\sum_{l=1}^{j} u_l \mathbb{1}_{\{0\le s\le \bar{t}_l\}}, \\ \tilde{F}_j(s,v,l') &:= F_j(s,v_1,\ldots,v_j) + \mathrm{i}\sigma_2(s)\mathbb{1}_{\{t_{l'-1}<s\le t_{l'}\}} v_{j+1}, \end{aligned} \tag{B.5}$$

$$
\begin{aligned}
M^{j,i}(u,T) &:= D^{j,i}(u,T)\ \exp\Big(\int_0^{\tilde t_i}\Big(\theta_s^1(E_{j,i}(s,u,T))+\theta_s^2(F_j(s,u))\Big)ds\Big)\ \widehat{f^j}(-u)\\
N^{j,i}(v,l',\kappa,T) &:= \tilde D^{j,i}(v-\mathrm{i}R,l',T)\ \exp\left(\int_0^{\tilde t_i}\Big(\theta_s^1(\tilde E_{j,i}(s,v-\mathrm{i}R,l',T))+\theta_s^2(\tilde F_j(s,v-\mathrm{i}R,l'))\Big)\,ds\right)\\
&\quad \times\widehat{f^j}(-v_1,\dots,-v_j)\ \frac{(1+\kappa)^{1-\mathrm{i}v_{j+1}-r}}{(\mathrm{i}v_{j+1}+r-1)(\mathrm{i}v_{j+1}+r)},\quad \text{for } v\in\mathbb{R}^{j+1}.
\end{aligned}
$$

Furthermore, we define for $\tilde t_i \le \tilde t_1$, $l' \le \ell(\tilde t_i)$, $0\le s\le \tilde t_i$ and $u\in\mathbb{R}$

$$
\begin{aligned}
\tilde E_0(s,u,l',\tilde t_i) &:= \Sigma(s,\tilde t_i)\\
&\quad +\mathrm{i}\left(\beta(s)\mathbb{1}_{\{t_{l'-1}<s\le t_{l'}\}}+\Sigma(s,t_{l'-1})\mathbb{1}_{\{0\le s\le t_{l'-1}\}}-\Sigma(s,t_{l'})\mathbb{1}_{\{0\le s\le t_{l'}\}}\right)u,\\
\tilde F_0(s,u,l') &:= \mathrm{i}\sigma_2(s)\mathbb{1}_{\{t_{l'-1}<s\le t_{l'}\}}u,\\
N_0(u,l',\kappa,\tilde t_i) &:= \exp\big((r+\mathrm{i}u)\widetilde{w}_{t_{l'}}\big)\ \exp\left(\int_0^{\tilde t_i}\Big(\theta_s^1(\tilde E_0(s,u-\mathrm{i}r,l',\tilde t_i))+\theta_s^2(\tilde F_0(s,u-\mathrm{i}r,l'))\Big)\,ds\right)\\
&\quad\times\frac{(1+\kappa)^{1-\mathrm{i}u-r}}{(\mathrm{i}u+r-1)(\mathrm{i}u+r)}.
\end{aligned}
$$

B.3. **Functions for Theorem 4.3.** For w_l from Equation (B.2), $0\le s\le T$, $i\in\{2,\dots,K-1\}$, $u\in\mathbb{R}^{i-1}$, and $v\in\mathbb{R}^i$, we define

$$
\begin{aligned}
D^i(u,T) &:= \exp\left(\mathrm{i}\sum_{l=1}^{i-1}u_l w_l-\omega(\bar t_i)\right),\\
\tilde D^i(v,T) &:= D^i(v_1,\dots,v_{i-1},T)\exp\big(\mathrm{i}\ v_i w_i\big),
\end{aligned}
$$

$$
\begin{aligned}
E^i(s,u,T) &:= \mathrm{i}(\beta(s)-\Sigma(s,T))\sum_{l=1}^{i-1}u_l\mathbb{1}_{\{0\le s\le \bar t_l\}}+\beta(s),\\
\tilde E^i(s,v,T) &:= E^i(s,v_1,\dots,v_{i-1},T)+\mathrm{i}(\beta(s)-\Sigma(s,T))v_i,
\end{aligned}
$$

$$
\begin{aligned}
F^i(s,u) &:= \mathrm{i}\sigma_2(s)\sum_{l=1}^{i-1}u_l\mathbb{1}_{\{0\le s\le \bar t_l\}}+\sigma_2(s),\\
\tilde F^i(s,v) &:= F^i(s,v_1,\dots,v_{i-1})+\mathrm{i}\sigma_2(s)v_i,
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
M^i(u,T) &:= D^i(u,T)\ \exp\Big(\int_0^{\bar t_i}\Big(\theta_s^1(E^i(s,u,T))+\theta_s^2(F^i(s,u))\Big)ds\Big)\ \widehat{f^{i-1}}(-u),\\
N^i(v,T) &:= \tilde D^i(v,T)\ \exp\Big(\int_0^{\bar t_i}\Big(\theta_s^1(\tilde E^i(s,v,T))+\theta_s^2(\tilde F^i(s,v))\Big)ds\Big)\ \widehat{f^i}(-v).
\end{aligned}
$$

Online Supplementary Material for Fourier based methods for the management of complex life insurance products

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Appendix C. Proof of Theorem 4.2

Proof. By definition,

$$\mathrm{P}^{\mathrm{DB}} = \sum_{i=1}^{N'} E_Q\left[e^{-\int_0^{\tilde{t}_i} r(u)du}\mathrm{DB}(\tilde{t}_i)\right].$$

As the mortality intensity is independent of the financial market, we obtain that

$$E_Q\left[e^{-\int_0^{\tilde{t}_i} r(u)du}\mathrm{DB}(\tilde{t}_i)\right] = Q(\tau^m(x) \in [\tilde{t}_{i-1}, \tilde{t}_i))E_Q\left[e^{-\int_0^{\tilde{t}_i} r(u)du}\mathbb{1}_{\{\tau^s \geq \tilde{t}_i\}}Z_{\tilde{t}_i}\right].$$

We distinguish two cases, $\tilde{t}_i \leq \bar{t}_1$ and $\bar{t}_1 < \tilde{t}_i$, and start with the detailed description of the second case.

For $j \in \{1, \ldots, K-2\}$ and i such that $\bar{t}_j < \tilde{t}_i \leq \bar{t}_{j+1}$, as well as for $j = K-1$ and i such that $\bar{t}_{K-1} < \tilde{t}_i \leq T$, we work along the same line as in the proof of Theorem 4.1, and get

$$\begin{aligned}
&E_Q\left[e^{-\int_0^{\tilde{t}_i} r(u)du}\mathbb{1}_{\{\tau^s \geq \tilde{t}_i\}}Z_{\tilde{t}_i}\right] \\
&= E_Q\left[e^{-\int_0^{\tilde{t}_i} r(u)du}\mathbb{1}_{\{\tau^s \geq \bar{t}_{j+1}\}}Z_{\tilde{t}_i}\right] = E_Q\left[e^{-\int_0^{\tilde{t}_i} r(u)du}e^{-\int_0^{\bar{t}_{j+1}} \lambda^s(u)du}Z_{\tilde{t}_i}\right] \\
&= I(1+\tilde{t}_i\varphi)E_Q\left[e^{-\int_0^{\tilde{t}_i} r(u)du}e^{-\int_0^{\bar{t}_{j+1}} \lambda^s(u)du}\right] \\
&+ I\sum_{l'=1}^{\ell(\tilde{t}_i)} E_Q\left[e^{-\int_0^{\tilde{t}_i} r(u)du}e^{-\int_0^{\bar{t}_{j+1}} \lambda^s(u)du}\left(\xi R_{t_{l'}} - \varphi\right)^+\right] \\
&- I\sum_{l'=1}^{\ell(\tilde{t}_i)} E_Q\left[e^{-\int_0^{\tilde{t}_i} r(u)du}e^{-\int_0^{\bar{t}_{j+1}} \lambda^s(u)du}\left(\xi R_{t_{l'}} - \gamma\right)^+\right].
\end{aligned} \tag{C.1}$$

The $\tilde{t}_i$-forward measure $Q^{\tilde{t}_i}$ is given by

$$\frac{dQ^{\tilde{t}_i}}{dQ} = \frac{1}{B(0,\tilde{t}_i)B(\tilde{t}_i)}. \tag{C.2}$$

Denoting the expectation with respect to $Q^{\tilde{t}_i}$ by $E^{\tilde{t}_i}$ the quantity in equation (C.1) is

$$\begin{aligned}
&IB(0,\tilde{t}_i)\Bigg((1+\tilde{t}_i\varphi)E^{\tilde{t}_i}\left[e^{-\int_0^{\bar{t}_{j+1}}\lambda^s(u)du}\right]+\xi\sum_{l'=1}^{\ell(\tilde{t}_i)}E^{\tilde{t}_i}\left[e^{-\int_0^{\bar{t}_{j+1}}\lambda^s(u)du}\left(R_{t_{l'}}-\frac{\varphi}{\xi}\right)^+\right]\\
&-\xi\sum_{l'=1}^{\ell(\tilde{t}_i)}E^{\tilde{t}_i}\left[e^{-\int_0^{\bar{t}_{j+1}}\lambda^s(u)du}\left(R_{t_{l'}}-\frac{\gamma}{\xi}\right)^+\right]\Bigg)\\
&=IB(0,\tilde{t}_i)\left((1+\tilde{t}_i\varphi)A^1_{j,i}+\xi\sum_{l'=1}^{\ell(\tilde{t}_i)}A^2_{j,i,l'}\left(\frac{\varphi}{\xi}\right)-\xi\sum_{l'=1}^{\ell(\tilde{t}_i)}A^2_{j,i,l'}\left(\frac{\gamma}{\xi}\right)\right),
\end{aligned}\tag{C.3}$$

with an obvious notation in the last line. We note that

$$e^{C(\bar{t}_{j+1}-\bar{t}_1)}A^1_{j,i}=E^{\tilde{t}_i}\Big[f^j(D(\bar{t}_1),\ldots,D(\bar{t}_j))\Big],\tag{C.4}$$

with $f^j(x_1,\ldots,x_j)=\prod_{l=1}^j f_l(x_l)$. As in (25), we obtain that the expectation in (C.4) can be represented as

$$\frac{1}{(2\pi)^j}\int_{\mathbb{R}^j}\tilde{M}^j_i(\mathrm{i}u)\widehat{f^j}(-u)du,\tag{C.5}$$

with

$$\widehat{f^j}(u_1,\ldots,u_j)=\prod_{l=1}^j\hat{f}_l(u_l),$$

and $\tilde{M}^j_i(\mathrm{i}u)$ defined as

$$\tilde{M}^j_i(\mathrm{i}u)=E^{\tilde{t}_i}\Big[e^{\mathrm{i}u_1D(\bar{t}_1)+\ldots+\mathrm{i}u_jD(\bar{t}_j)}\Big].$$

In virtue of (27), the Radon-Nikodym density (C.2) has explicit representation

$$\frac{dQ^{\tilde{t}_i}}{dQ}=\exp\Big(-\int_0^{\tilde{t}_i}A(s,\tilde{t}_i)ds+\int_0^{\tilde{t}_i}\Sigma(s,\tilde{t}_i)dL^1_s\Big).\tag{C.6}$$

Consequently, it follows from (21),

$$\begin{aligned}
\tilde{M}^j_i(\mathrm{i}u)&=E_Q\Big[e^{\mathrm{i}u_1D(\bar{t}_1)+\ldots+\mathrm{i}u_jD(\bar{t}_j)-\int_0^{\tilde{t}_i}A(s,\tilde{t}_i)ds+\int_0^{\tilde{t}_i}\Sigma(s,\tilde{t}_i)dL^1_s}\Big]\\
&=\exp\Big(\mathrm{i}\sum_{l=1}^j u_lw_l-\int_0^{\tilde{t}_i}A(s,\tilde{t}_i)ds\Big)\\
&\quad\times E_Q\Big[\exp\Big(\mathrm{i}\sum_{l=1}^j\Big(\int_0^{\bar{t}_l}u_l\sigma_2(s)dL^2_s+\int_0^{\bar{t}_l}u_l(\beta(s)-\Sigma(s,T))dL^1_s\Big)+\int_0^{\tilde{t}_i}\Sigma(s,\tilde{t}_i)dL^1_s\Big)\Big].
\end{aligned}$$

This last expectation is well defined in virtue of (4), and can be written as

$$\begin{aligned}
&E_Q\Big[\exp\Big(\int_0^{\tilde{t}_i}E_{j,i}(s,u,T)dL^1_s+\int_0^{\tilde{t}_i}F_j(s,u)dL^2_s\Big)\Big]\\
&\qquad=\exp\Big(\int_0^{\tilde{t}_i}\Big(\theta^1_s(E_{j,i}(s,u,T))+\theta^2_s(F_j(s,u))\Big)ds\Big),
\end{aligned}$$

due to (B.5) and equation (A.2). Therefore, with $D^{j,i}(u,T)$ defined in (B.5) we have

$$\tilde{M}^j_i(\mathrm{i}u)=D^{j,i}(u,T)e^{-\int_0^{\tilde{t}_i}A(s,\tilde{t}_i)ds}\exp\Big(\int_0^{\tilde{t}_i}\Big(\theta^1_s(E_{j,i}(s,u,T))+\theta^2_s(F_j(s,u))\Big)ds\Big).$$

Finally, combining (C.4) with (C.5) and the definition of $M^{j,i}(u,T)$ in (B.5), we deduce that

$$A^1_{j,i}=\frac{e^{-C(\bar{t}_{j+1}-\bar{t}_1)}}{(2\pi)^j}e^{-\int_0^{\tilde{t}_i}A(s,\tilde{t}_i)ds}\int_{\mathbb{R}^j}M^{j,i}(u,T)du.$$

Now we turn our attention to $A^2_{j,i,l'}(\kappa)$. We note that

$$e^{C(\bar{t}_{j+1}-\bar{t}_1)}A^2_{j,i,l'}(\kappa) = E^{\tilde{t}_i}\left[h^{j+1}\big(D(\bar{t}_1),\dots,D(\bar{t}_j),Y_{t_{l'}}-Y_{t_{l'-1}}\big)\right], \tag{C.7}$$

for $h^{j+1}(x_1,\dots,x_{j+1}) := f^j(x_1,\dots,x_j)(e^{x_{j+1}}-1-\kappa)^+$, with $f^j(\cdot)$ given above. In order to ensure integrability, let us define $H(x_1,\dots,x_{j+1}) := h^{j+1}(x_1,\dots,x_{j+1})e^{-rx_{j+1}}$, for some $1<r<2$, and

$$H_{j+1}(x_{j+1}) := (e^{x_{j+1}}-1-\kappa)^+e^{-rx_{j+1}}.$$

Then, H_{j+1} as well as H are integrable. Moreover, elementary integration shows that for all $y\in\mathbb{R}$

$$\hat{H}_{j+1}(y) = \frac{(1+\kappa)e^{(\mathrm{i}y-r)\log(1+\kappa)}}{(\mathrm{i}y-r+1)(\mathrm{i}y-r)} = \frac{(1+\kappa)^{\mathrm{i}y-r+1}}{(\mathrm{i}y-r+1)(\mathrm{i}y-r)}.$$

Observe that $|\hat{H}_{j+1}(y)|_{\mathbb{C}} = (1+\kappa)e^{-r\log(1+\kappa)}(((1-r)^2+y^2)(r^2+y^2))^{-1/2}$, thus, $\hat{H}_{j+1}$ is integrable. Therefore, combining the last result with the integrability of $\widehat{f^j}$, we deduce that $\hat{H}$ is integrable, and

$$\hat{H}(y_1,\dots,y_{j+1}) = \widehat{f^j}(y_1,\dots,y_j)\,\frac{(1+\kappa)^{\mathrm{i}y_{j+1}-r+1}}{(\mathrm{i}y_{j+1}-r+1)(\mathrm{i}y_{j+1}-r)}. \tag{C.8}$$

Therefore, it follows from Theorem 3.2 in Eberlein et al. (2010) that

$$E^{\tilde{t}_i}\left[h^{j+1}\big(D(\bar{t}_1),\dots,D(\bar{t}_j),Y_{t_{l'}}-Y_{t_{l'-1}}\big)\right] = \frac{1}{(2\pi)^{j+1}}\int_{\mathbb{R}^{j+1}}\tilde{N}^{j+1}_{i,l'}(R+\mathrm{i}u)\widehat{h^{j+1}}(\mathrm{i}R-u)du, \tag{C.9}$$

for $R=(0,\dots,0,r)\in\mathbb{R}^{j+1}$, $1<r<2$, and $\tilde{N}^{j+1}_{i,l'}(R+\mathrm{i}u)$ defined as

$$\tilde{N}^{j+1}_{i,l'}(R+\mathrm{i}u) := E^{\tilde{t}_i}\left[e^{\mathrm{i}u_1D(\bar{t}_1)+\ldots+\mathrm{i}u_jD(\bar{t}_j)+(\mathrm{i}u_{j+1}+r)(Y_{t_{l'}}-Y_{t_{l'-1}})}\right]. \tag{C.10}$$

Consequently, using (C.6), we have

$$\begin{aligned}
&\tilde{N}^{j+1}_{i,l'}(R+\mathrm{i}u)\\
&= E_Q\left[e^{\mathrm{i}u_1D(\bar{t}_1)+\ldots+\mathrm{i}u_jD(\bar{t}_j)+(\mathrm{i}u_{j+1}+r)(Y_{t_{l'}}-Y_{t_{l'-1}})-\int_0^{\tilde{t}_i}A(s,\tilde{t}_i)ds+\int_0^{\tilde{t}_i}\Sigma(s,\tilde{t}_i)dL^1_s}\right]\\
&= \exp\Big(\mathrm{i}\sum_{l=1}^{j}u_lw_l+(\mathrm{i}u_{j+1}+r)\widetilde{w}_{t_{l'}}-\int_0^{\tilde{t}_i}A(s,\tilde{t}_i)ds\Big)\\
&\quad\times E_Q\Bigg[\exp\Big(\mathrm{i}\sum_{l=1}^{j}\Big(\int_0^{\bar{t}_l}u_l\sigma_2(s)dL^2_s+\int_0^{\bar{t}_l}u_l(\beta(s)-\Sigma(s,T))dL^1_s\Big)\\
&\qquad+(\mathrm{i}u_{j+1}+r)\Big(\int_{]t_{l'-1},t_{l'}]}\sigma_2(s)dL^2_s+\int_{]t_{l'-1},t_{l'}]}\beta(s)dL^1_s\\
&\qquad-\int_0^{t_{l'}}\Sigma(s,t_{l'})dL^1_s+\int_0^{t_{l'-1}}\Sigma(s,t_{l'-1})dL^1_s\Big)+\int_0^{\tilde{t}_i}\Sigma(s,\tilde{t}_i)dL^1_s\Big)\Bigg].
\end{aligned}$$

Using the definitions from (B.5), the above can be rewritten as

$$\begin{aligned}
\tilde{N}^{j+1}_{i,l'}(R+\mathrm{i}u) \;=\;& \tilde{D}^{j,i}(u-\mathrm{i}R,l',T)e^{-\int_0^{\tilde{t}_i}A(s,\tilde{t}_i)ds}\\
&\times E_Q\Big[\exp\Big(\int_0^{\tilde{t}_i}\tilde{E}_{j,i}(s,u-\mathrm{i}R,l',T)dL^1_s+\int_0^{\tilde{t}_i}\tilde{F}_j(s,u-\mathrm{i}R,l')dL^2_s\Big)\Big].
\end{aligned}$$

Observe that, due to $1<r<2$, as well as (3) and (4), $r\sigma_2(s)\le M_2$ and $|r\beta(s)-r\Sigma(s,t_{l'})+r\Sigma(s,t_{l'-1})+\Sigma(s,\tilde{t}_i)|\le 6M_1/7+M_1/7=M_1$. Thus, using the independence of the driving processes and (A.2), the above expectation can be explicitly calculated

$$\begin{aligned}
\tilde{N}^{j+1}_{i,l'}(R+\mathrm{i}u) \;=\;& \tilde{D}^{j,i}(u-\mathrm{i}R,l',T)e^{-\int_0^{\tilde{t}_i}A(s,\tilde{t}_i)ds}\\
&\times\exp\Big(\int_0^{\tilde{t}_i}\theta^1_s(\tilde{E}_{j,i}(s,u-\mathrm{i}R,l',T))ds+\int_0^{\tilde{t}_i}\theta^2_s(\tilde{F}_j(s,u-\mathrm{i}R,l'))ds\Big).
\end{aligned}\tag{C.11}$$

On the other hand, we observe that for any $u \in \mathbb{R}^{j+1}$,

$$\hat{H}(u) = \int_{\mathbb{R}^{j+1}} e^{\mathrm{i}\langle u,x\rangle} e^{-\langle R,x\rangle} h^{j+1}(x)dx = \widehat{h^{j+1}}(u + \mathrm{i}R).$$

Consequently, we deduce that

$$\widehat{h^{j+1}}(\mathrm{i}R - u) = \hat{H}(-u) = \widehat{f^j}(-u_1, \dots, -u_j)\ \frac{(1+\kappa)^{1-\mathrm{i}u_{j+1}-r}}{(\mathrm{i}u_{j+1} + r - 1)(\mathrm{i}u_{j+1} + r)}. \tag{C.12}$$

Plugging (C.11) and (C.12) in (C.9), and using the definition of $N^{j,i}(u, l', \kappa, T)$ in (B.5), it follows that

$$A^2_{j,i,l'}(\kappa) = \frac{e^{-C(\bar{t}_{j+1}-\bar{t}_1)}}{(2\pi)^{j+1}} e^{-\int_0^{\tilde{t}_i} A(s,\tilde{t}_i)ds} \int_{\mathbb{R}^{j+1}} N^{j,i}(u, l', \kappa, T)du.$$

Now we consider the case $\tilde{t}_i \le \bar{t}_1$ for $i \in \{1, \dots, N'\}$. Observe that

$$E_Q\left[e^{-\int_0^{\tilde{t}_i} r(u)du} \mathbb{1}_{\{\tau^s \ge \tilde{t}_i\}} Z_{\tilde{t}_i}\right] = E_Q\left[e^{-\int_0^{\tilde{t}_i} r(u)du} Z_{\tilde{t}_i}\right],$$

from which it follows that

$$\begin{aligned}
&E_Q\left[e^{-\int_0^{\tilde{t}_i} r(u)du} \mathbb{1}_{\{\tau^s \ge \tilde{t}_i\}} Z_{\tilde{t}_i}\right] \\
&= IB(0,\tilde{t}_i)\Big((1+\tilde{t}_i\varphi) + \sum_{l'=1}^{\ell(\tilde{t}_i)} E^{\tilde{t}_i}\Big[\Big(\xi R_{t_{l'}} - \varphi\Big)^+\Big] - \sum_{l'=1}^{\ell(\tilde{t}_i)} E^{\tilde{t}_i}\Big[\Big(\xi R_{t_{l'}} - \gamma\Big)^+\Big]\Big) \\
&= IB(0,\tilde{t}_i)\left((1+\tilde{t}_i\varphi) + \xi \sum_{l'=1}^{\ell(\tilde{t}_i)} \left(A_{i,l'}\left(\frac{\varphi}{\xi}\right) - A_{i,l'}\left(\frac{\gamma}{\xi}\right)\right)\right),
\end{aligned}$$

with the definition

$$A_{i,l'}(\kappa) = E^{\tilde{t}_i}\left[\left(R_{t_{l'}} - \kappa\right)^+\right] = E^{\tilde{t}_i}\left[h_1(Y_{t_{l'}} - Y_{t_{l'-1}})\right]$$

for the function $h_1(x) = (e^x - 1 - \kappa)^+$. In order to enforce the integrability of h_1 we consider for some $1 < r < 2$ the dampened function $H_1(x) = h_1(x)e^{-rx}$, and apply Theorem 2.2 in Eberlein et al. (2010) to get

$$E^{\tilde{t}_i}\big[h_1\big(Y_{t_{l'}} - Y_{t_{l'-1}}\big)\big] = \frac{1}{2\pi}\int_{\mathbb{R}} N^i_{l'}(r + \mathrm{i}u)\hat{h}_1(\mathrm{i}r - u)du, \tag{C.13}$$

with

$$N^i_{l'}(r + \mathrm{i}u) := E^{\tilde{t}_i}\big[e^{(r+\mathrm{i}u)(Y_{t_{l'}} - Y_{t_{l'-1}})}\big].$$

Now we use the explicit forms of the density $\frac{dQ^{\tilde{t}_i}}{dQ}$ and the increment $Y_{t_{l'}} - Y_{t_{l'-1}}$ (see proof of Theorem 4.1) to obtain

$$\begin{aligned}
N^i_{l'}(r + \mathrm{i}u) \ = \ & \exp\Big((r+\mathrm{i}u)\widetilde{w}_{t_{l'}} - \int_0^{\tilde{t}_i} A(s,\tilde{t}_i)ds\Big) \\
& \times E_Q\Big[\exp\Big(\int_0^{\tilde{t}_i} \Sigma(s,\tilde{t}_i)dL^1_s + (r+\mathrm{i}u)\Big(\int_{]t_{l'-1},t_{l'}]} \sigma_2(s)dL^2_s + \int_{]t_{l'-1},t_{l'}]} \beta(s)dL^1_s \\
& - \int_0^{t_{l'}} \Sigma(s,t_{l'})dL^1_s + \int_0^{t_{l'-1}} \Sigma(s,t_{l'-1})dL^1_s\Big)\Big)\Big]
\end{aligned}$$

from which it follows that

$$\begin{aligned} N^i_{l'}(r+\mathrm{i}u) &= \exp\Big((r+\mathrm{i}u)\widetilde{w}_{t_{l'}} - \int_0^{\tilde{t}_i} A(s,\tilde{t}_i)ds\Big) \\ &\quad \times E_Q\Big[\exp\Big(\int_0^{\tilde{t}_i} \tilde{E}_0(s,u-\mathrm{i}r,l',\tilde{t}_i)dL^1_s + \int_0^{\tilde{t}_i} \tilde{F}_0(s,u-\mathrm{i}r,l')dL^2_s\Big)\Big]. \end{aligned}$$

The last expectation can be computed using formula (A.2), which returns

$$\exp\Big(\int_0^{\tilde{t}_i} \theta^1_s(\tilde{E}_0(s,u-\mathrm{i}r,l',\tilde{t}_i))ds + \int_0^{\tilde{t}_i} \theta^2_s(\tilde{F}_0(s,u-\mathrm{i}r,l'))ds\Big).$$

As

$$\hat{h}_1(\mathrm{i}r-u) = \frac{(1+\kappa)^{1-\mathrm{i}u-r}}{(\mathrm{i}u+r-1)(\mathrm{i}u+r)}, \tag{C.14}$$

we finally obtain

$$A_{i,l'}(\kappa) = \frac{e^{-\int_0^{\tilde{t}_i} A(s,\tilde{t}_i)ds}}{2\pi} \int_{\mathbb{R}} N_0(u,l',\kappa,\tilde{t}_i)du.$$

□

Appendix D. Analytic expression of the survival probability

In reference to the stochastic mortality model set up in Sections 3.2 and 5, the relevant expressions for the survival probability in (13) are

$$\begin{aligned} A_x(t) &:= \frac{c_1\exp(c_2t)}{c_3(c_2+c_3)}[1-\exp(-(c_2+c_3)t)] + \frac{1}{4}\Big(\frac{c_4}{c_3}\Big)^2\frac{\exp(2c_5t)}{c_5}[1-\exp(-2c_5t)] \\ &\quad -\Big(\frac{c_4}{c_3}\Big)^2\frac{\exp(2c_5t)}{2c_5+c_3}[1-\exp(-(2c_5+c_3)t)] - \frac{c_1\exp(c_2t)}{c_2c_3}[1-\exp(-c_2t)] \\ &\quad +\frac{1}{4}\Big(\frac{c_4}{c_3}\Big)^2\frac{\exp(2c_5t)}{c_3+c_5}[1-\exp(-2(c_3+c_5)t)], \\ B_x(t) &:= \frac{1}{c_3}[\exp(-c_3t)-1], \end{aligned}$$

and

$$c_1 := \frac{\kappa}{q}\exp\Big(\frac{x-z}{q}\Big), c_2 := \frac{1}{q}-\lambda, c_3 := \kappa-\frac{1}{q}, c_4 := \frac{\sigma}{q}\exp\Big(\frac{x-z}{q}\Big), c_5 := \frac{1}{q},$$

(see Escobar et al., 2016).

Appendix E. Further numerical results

In order to perform a sensible comparison, we consider a 5 year DB, and a 4 year SB. Due to the number of terms involved in the computation of the DB, we only consider the terms in the second summation of Theorem 4.2. Concerning the SB, we note that the first term in the sum defining P^{SB} in Theorem 4.3 is composed by a constant (B^1_1), and a one-dimensional integral (B^2_1), which is obtained by deterministic quadrature and therefore is not considered in this benchmarking exercise. Results are reported in Table E.1.

Table E.1. Benchmarking Monte Carlo integration with importance sampling: DB: maturity $T = 5$ years; SB: maturity $T = 4$ years. Cap rate: $\gamma = 0.05$, participation rate $\xi = 0.7$. Other parameters: Table 1. 'Quadrature': Matlab built-in functions `integral`, and `integral2`. 'MC': Monte Carlo integration. 'MCi': Monte Carlo integration with interpolation. Bias/standard error expressed as percentage of the actual value. Monte Carlo iterations: 100 batches of size 10^6.

DB ($T = 5$ years)

A. Intensity 1

	Quadrature	MC	bias %	std. err. %	MCi	bias %	std. err. %
$A^1_{1,2}$	0.9865	0.9864	0.0071	0.0057	0.9864	0.0084	0.0056
$A^2_{1,2,1}(\varphi/\xi)$	0.0887	0.0886	0.0663	0.1548	0.0886	0.0306	0.1560
$A^2_{1,2,2}(\varphi/\xi)$	0.0888	0.0888	0.0699	0.1558	0.0888	0.0236	0.1569
$A^2_{1,2,1}(\gamma/\xi)$	0.0667	0.0666	0.1132	0.2048	0.0667	0.0703	0.2063
$A^2_{1,2,2}(\gamma/\xi)$	0.0669	0.0668	0.0627	0.2059	0.0669	0.0083	0.2074

B. Intensity 2

	Quadrature	MC	bias %	std. err. %	MCi	bias %	std. err. %
$A^1_{1,2}$	0.9797	0.9797	0.0009	0.0007	0.9797	0.0006	0.0007
$A^2_{1,2,1}(\varphi/\xi)$	0.0880	0.0880	0.0319	0.1156	0.0884	0.4314	0.1135
$A^2_{1,2,2}(\varphi/\xi)$	0.0882	0.0882	0.0207	0.1153	0.0882	0.0148	0.1139
$A^2_{1,2,1}(\gamma/\xi)$	0.0661	0.0661	0.0476	0.1632	0.0664	0.4883	0.1602
$A^2_{1,2,2}(\gamma/\xi)$	0.0664	0.0664	0.0320	0.1625	0.0664	0.0167	0.1605

SB ($T = 4$ years)

A. Intensity 1

	Quadrature	MC	bias %	std. err. %	MCi	bias %	std. err. %
B^1_2	0.9871	0.9872	0.0030	0.0031	0.9872	0.0029	0.0031
B^2_2	0.9717	0.9718	0.0046	0.0047	0.9718	0.0047	0.0047

B. Intensity 2

	Quadrature	MC	bias %	std. err. %	MCi	bias %	std. err. %
B^1_2	0.9806	0.9806	0.00001	0.0007	0.9806	0.00002	0.0007
B^2_2	0.9579	0.9578	0.0007	0.0012	0.9579	0.0007	0.0012