Bid and Ask Prices as Non-Linear Continuous Time G-Expectations Based on Distortions

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Abstract

Probability distortions for constructing nonlinear G-expectations for the bid and ask or lower and upper prices in continuous time are here extended to the direct use of measure distortions. Fairly generally measure distortions can be constructed as probability distortions applied to an exponential distribution function on the half line. The valuation methodologies are extended beyond contract valuation to the valuation of economic activities with infinite lives. Explicit computations illustrate the procedures for stock indices and insurance loss processes.

Keywords: Discounted Variance Gamma, Measure Distortions, Inhomogeneous Loss Process, Law invariant risk measures

1 Introduction

Asset pricing in liquid financial markets has developed the theory of risk neutral valuation. Based on principles of no arbitrage discounted prices for claims with no intermediate cash flows are seen to be martingales under a suitably chosen equilibrium pricing probability. The martingale condition in Markovian contexts reduces the pricing problem to an equivalent solution of a linear partial differential or integro-differential equation subject to a boundary condition at maturity. The essential property of market liquidity is the supposition of the law of one price or the ability, on the part of market participants, to trade in both directions at the same price.

In the absence of such liquidity, the law of one price is abandoned and we get at a minimum a two price economy where the terms of trade depend on the direction of trade. Such a two price equilibrium was studied in a static one period context in Madan (2012). The two equilibrium prices arise on account of

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an exposure to residual risk that cannot be eliminated, by construction, and the prices are designed to make this exposure acceptable. Acceptability is modeled by requiring positive expectations under a whole host of test or scenario probabilities as described for example in Artzner, Delbaen, Eber and Heath (1999). As a consequence the ask or upper price turns out to be the supremum of test valuations while the bid or lower price is an infimum of the same set of test valuations. The resulting pricing operators are now nonlinear on the space of random variables, with the lower price being concave and the upper price convex. In particular the upper price of a package of risks is smaller than the sum of component prices while the lower price is similarly above.

When the decision of risk acceptability is further modeled as solely depending on the probability distribution of the risk and if in addition we ask for additivity of the two prices, for risks that are monotonically related, then closed forms for the two prices become available (see Kusuoka (2001)). Specifically, the lower price may be expressed as an expectation computed after distorting the risk distribution function by composing it with a prespecified concave distribution function on the unit interval. Such a formulation was proposed and tested on option market data in Cherny and Madan (2010). Carr, Madan and Vicente Alvarez (2011) employ this approach to define capital requirements and up front profits on trades. Eberlein and Madan (2012) apply the method to estimate capital requirements for the financial sector during the financial crisis of 2008.

Dynamically consistent two price sequences based on locally applying probability distortions are examples of non-linear expectations as studied in Cohen and Elliott (2010). Madan and Schoutens (2012b) apply such pricing principles to study the impact of illiquidity on a variety of financial markets. The lower price is a submartingale while the upper price is a supermartingale with the two prices converging to each other and the payout at maturity. Madan, Wang and Heckman (2011) apply discrete time distortion based nonlinear expectations to the valuation of insurance liabilities.

In continuous time the two prices are related to nonlinear expectations seen as the G-expectations introduced by Peng (2004). Probability distortions were used to formulate G-expectations for the upper and lower price in Eberlein, Madan, Pistorius, Schoutens and Yor (2012). This paper extends the theory of distortion based G-expectations in two directions. The first is to generalize away from probability distortions to measure distortions as they arose in Madan, Pistorius and Stadje (2013) where the continuous time limit of discrete time distortion based nonlinear expectations was investigated. Here we directly introduce and apply measure distortions. The second extension deals with the convergence of bid ask spreads to zero at maturity. Though many contracts have explicit maturities, economic activities of running airlines, insuring losses, selling goods and services need to be, and are valued, in financial markets with no apparent maturity. Madan and Yor (2012) introduce valuation models for such claims termed stochastic perpetuities conducted under a liquid, law of one price setting. The resulting martingales are uniformly integrable and the explicit maturity is transferred to infinity. Here we extend distortion based G-expectations to valuation processes with an infinite maturity.

The theory is illustrated on the two price valuation of stocks. We employ both the quadratic variation based probability distortion introduced in Eberlein, Madan, Pistorius, Schoutens and Yor (2012) and the new measure based distortion introduced here. We also apply measure distortions to value compound Poisson processes of insurance loss liabilities for both the homogeneous and inhomogeneous cases.

The outline of the rest of the paper is as follows. Section 2 introduces measure distortions for the distortion of jump compensators of Lévy systems. Section 3 summarizes details of the discounted variance gamma stock valuation model of Madan and Yor (2012). The quadratic variation based probability distortion is then summarized and applied to construct lower and upper prices for the discounted variance gamma model in section 4. Specific measure distortions are introduced in Section 5 and applied to the discounted variance gamma model. Section 6 presents the use of measure distortions in the two price valuation of insurance loss processes. Section 7 comments on the design of measure distortions. Section 9 presents a conjectured solution in the case of inhomogeneous compound Poisson losses. Section 9 concludes.

2 Measure distortions

This section introduces the use of measure distortions for defining the acceptability of a set of random variables in the context of a static one period model. In continuous time we apply this structure locally to instantaneous risk characteristics. Consider first the acceptability of a random variable X with distribution function $F(x)$. When acceptability is defined just in terms of the distribution function it may be reduced to a positive expectation under a fixed concave distortion. More precisely, let Ψ be a fixed concave distribution function defined on the unit interval. The random variable X is acceptable just if the expectation of X taken with respect to the distorted distribution function $\Psi(F(x))$ is nonnegative, or

$$\mathcal{E}(X) = \int_{-\infty}^{\infty} x d\Psi(F(x)) \geq 0, \tag{1}$$

where $\mathcal{E}(X)$ refers to a distorted expectation. X is strictly acceptable when $\mathcal{E}(X) > 0$. Nonnegative expectations under concave distortions have been used to define acceptable risks in Cherny and Madan (2009, 2010) with dynamic extensions being recently developed by Bielecki, Cialenco and Zhang (2011). See also Madan (2010), Eberlein and Madan (2012), Carr, Madan and Vicente Alvarez (2011) and Madan and Schoutens (2011a, 2011b, 2012a, 2012b) for other applications. Wang, Young and Panjer (1997) provide an axiomatic characterization of insurance prices using such distorted expectations.

After splitting the distorted expectation (1) at zero and integrating by parts we may write the distorted expectation as a Choquet integral in the form

$$\mathcal{E}(X) = \int_{0}^{\infty} (1 - \Psi(F(x)))dx - \int_{-\infty}^{0} \Psi(F(x))dx. \tag{2}$$

It is now useful to follow Madan, Pistorius and Stadje (2012) and introduce

$$\widehat{F}(x) = 1 - F(x) \tag{3}$$

and

$$\widehat{\Psi}(x) = 1 - \Psi(1 - x) \tag{4}$$

and write the distorted expectation (2) as

$$\mathcal{E}(X) = \int_0^\infty \widehat{\Psi}(\widehat{F}(x))dx - \int_{-\infty}^0 \Psi(F(x))dx. \tag{5}$$

Measure distortions will follow from expression (5). But first we connect these expressions to bid and ask prices or lower and upper prices.

Formally, Artzner, Delbaen, Eber and Heath (1999) show that acceptable random variables form a convex cone and earn their acceptability by having a positive expectation under a set $\mathcal{M}$ of scenario or test measures equivalent to the original probability P. Cherny and Madan (2010) then introduce the bid, $b(X)$, and ask, $a(X)$ or upper or lower prices as

$$\begin{aligned} b(X) &= \inf_{Q \in \mathcal{M}} E^Q[X] \\ a(X) &= \sup_{Q \in \mathcal{M}} E^Q[X] \end{aligned}$$

with acceptability being equivalent to $b(X) \geq 0$. As a consequence $b(X) = \mathcal{E}(X)$. The set of measures Q supporting acceptability or the set $\mathcal{M}$ is identified in Madan, Pistorius and Stadje (2012) as all measures Q, absolutely continuous with respect to P, with square integrable densities, that satisfy for all sets A, the condition

$$\widehat{\Psi}(P(A)) \leq Q(A) \leq \Psi(P(A)). \tag{6}$$

We refer the reader to Madan, Pistorius and Stadje (2012) and the references cited therein for the proof of this result, but offer here some intuition driving these probability bounds.

Consider the lottery $\mathbf{1}_A$ being sold at price $Q(A)$. The empirical distribution function for the buyer of this lottery has a probability $1 - P(A)$ of the payoff $-Q(A)$ and a probability $P(A)$ for the payoff $1 - Q(A)$. The price $Q(A)$ rules out strict acceptability if the distorted expectation

$$-Q(A)\Psi(1 - P(A)) + (1 - Q(A))(1 - \Psi(1 - P(A)) \leq 0$$

or equivalently that the lower bound holds or

$$\widehat{\Psi}(P(A)) \leq Q(A).$$

Similarly the empirical distribution function for the seller has a probability $P(A)$ of the payoff $-1 + Q(A)$ and a probability $1 - P(A)$ of the payoff $Q(A)$. Strict acceptability is avoided by the price $Q(A)$ if

$$(-1 + Q(A))\Psi(P(A)) + Q(A)(1 - \Psi(P(A)) \leq 0$$

or equivalently that

$$Q(A) \leq \Psi(P(A)).$$

The probability bounds (6) rule out the strict acceptability of buying or selling simple lotteries.

We now remark on the distortion Ψ and its complementary distortion $\widehat{\Psi}$. Both distortions are monotone increasing in their arguments but Ψ is concave and bounded below the identity function while $\widehat{\Psi}$ is convex and bounded above by the identity function. For the bid price or the distorted expectation one employs the concave distortion on the losses or negative outcomes while one employs the convex distortion on the gains or positive outcomes. This is reasonable as distorted expectations are expectations under a change of measure with the measure change being the derivative of the distortion taken at the quantile. The concave distortion then reweights upwards the lower quantiles associated with large losses, while the convex distortion reweights downward upper tail. This structural reweighting will be maintained on passage to measure distortions.

Consider now in place of an expectation an integration with respect to a positive, possibly infinite measure μ or the measure integral

$$m = \int_{-\infty}^{\infty} v(y)\mu(dy) < \infty. \tag{7}$$

Though the measure may be infinite, we suppose that all the tail measures are finite. We may then rewrite the measure integral (7) as

$$m = -\int_{-\infty}^{0} \mu\left((v \leq x)\right) dx + \int_{0}^{\infty} \mu\left((v > x)\right) dx. \tag{8}$$

We now consider two functions Γ_+, Γ_- defined on the positive half line that are zero at zero, monotone increasing, respectively concave and convex, and respectively bounded below and above by the identity function. These functions will now be used to distort the measure μ and we refer to them as measure distortions. We then define the distorted measure integral as

$$\mathfrak{m} = -\int_{-\infty}^{0} \Gamma_+\left(\mu(v \leq x)\right) dx + \int_{0}^{\infty} \Gamma_-\left(\mu\left(v > x\right)\right) dx, \tag{9}$$

where we assume both integrals are finite.

For computational purposes we shall employ

$$\mathfrak{m} = \int_{-\infty}^{0} xd\left(\Gamma_+(\mu(v \leq x))\right) - \int_{0}^{\infty} xd\left(\Gamma_-\left(\mu\left(v > x\right)\right)\right) \tag{10}$$

Acceptability of a random outcome with respect to a possibly infinite measure with finite tail measures may then be defined by a positive distorted measure integral. Madan, Pistorius and Stadje (2012) identify the set of supporting measures as absolutely continuous with respect to μ with square integrable densities that satisfy for all sets A, for which $\mu(A) < \infty$ the condition

$$\Gamma_-(\mu(A)) \leq Q(A) \leq \Gamma_+\left(\mu(A)\right).$$

We shall replace measure integrals of the form (7) by distorted measure integrals (9) in defining G-expectations as solutions of nonlinear partial integro-differential equations.

3 The discounted variance gamma model

This section introduces the discounted variance gamma (dvg) model of Madan and Yor (2012) as the driving uncertainty for the stock price. The discounted stock price is modeled as a positive martingale on the positive half line. The discounted stock price responds to positive and negative shocks given by two independent gamma processes. The variance gamma model of Madan and Seneta (1990), Madan, Carr and Chang (1998) has such a representation as the difference of two independent gamma processes, but unlike the variance gamma process, as we now consider perpetuities, the shocks are discounted in their effects on the discounted stock price. More specifically, let $\gamma_p(t)$ and $\gamma_n(t)$ be two independent standard gamma processes (with unit scale and shape parameters) and define for an interest rate r the process

$$X\left(t\right) = \int_0^t b_p e^{-rs} d\gamma_p(c_p s) - \int_0^t b_n e^{-rs} d\gamma_n(c_n s).$$

The parameters $b_p > 0, c_p > 0$ and $b_n > 0, c_n > 0$ reflect the scale and shape parameters of the undiscounted gamma processes, however, $X(t)$ accumulates discounted shocks. The characteristic function for $X(t)$ is explicitly derived in Madan and Yor (2012) and is shown to be

$$E\left[\exp\left(iuX(t)\right)\right] = \exp\left(\begin{array}{c} \frac{c_p}{r}\left(dilog\left(iub_p\right) - dilog\left(iub_p e^{-rt}\right)\right) \\ +\frac{c_n}{r}\left(dilog\left(-iub_n\right) - dilog\left(-iub_n e^{-rt}\right)\right) \end{array}\right) \tag{11}$$

where the *dilog* function is given by

$$dilog(x) = -\int_0^x \frac{\ln(1-t)}{t} dt. \tag{12}$$

The discounted stock price driven by the discounted variance gamma process is given by the positive martingale

$$M(t) = \exp\left(X(t) + \omega(t)\right) \tag{13}$$

where

$$\exp\left(\omega(t)\right) = \frac{1}{E\left[\exp\left(X(t)\right)\right]}.$$

Unlike geometric Brownian motion or exponential Lévy models, the martingale (13) is uniformly integrable on the half line and the discounted stock price at infinity is a well defined positive random variable

$$M\left(\infty\right) = \exp\left(X(\infty) + \omega(\infty)\right) \tag{14}$$

where

$$X(\infty) = \int_0^\infty b_p e^{-rs} d\gamma_p(c_p s) - \int_0^\infty b_n e^{-rs} d\gamma_n(c_n s) \tag{15}$$

and

$$E\left[\exp\left(iuX(\infty)\right)\right] = \exp\left(\frac{c_p}{r} dilog\left(iub_p\right) + \frac{c_n}{r} dilog\left(-iub_n\right)\right). \tag{16}$$

Consider now any claim promising at infinity the payout in time zero dollars of $F(M(\infty))$. Equivalently one may consider the limit as T goes to infinity of the claim paying at T, the sum $e^{rT}F(M(T))$. Markets in the future and hence markets at all times t price the claim in time zero dollars at the risk neutral price of

$$w_F(t) = E\left[F(M(\infty))|\mathcal{F}_t\right]. \tag{17}$$

By construction the price process $w_F(t)$ is a martingale.

Let Y be an independent random variable with the same law as that of $X(\infty)$. To determine the price $w_F(t)$ we note that

$$\begin{aligned} X(\infty) &= X(t) + \int_t^\infty b_p e^{-ru} d\gamma_p(c_p u) - \int_t^\infty b_n e^{-ru} d\gamma_n(c_n u) \\ &\overset{(d)}{=} X(t) + e^{-rt} Y. \end{aligned} \tag{18}$$

We thus observe that conditional on t, there is a function $H(X, v)$, with $v = e^{-rt}$ such that

$$w_F(t) = H(X(t), e^{-rt}). \tag{19}$$

The martingale condition on $w_F(t)$ then implies that

$$-rvH_v + \int_{-\infty}^{\infty} \left(H(X+y, v) - H(X, v)\right) k(y, v) dy = 0 \tag{20}$$

where $k(y, v)$ is the Lévy system associated with the jumps of the process $X(t)$.

The price process is determined on solving the partial integro-differential equation (20), subject to the boundary condition

$$H(X(\infty), 0) = F(\exp\left(X(\infty) + \omega(\infty)\right)) \tag{21}$$

in the interval $0 \leq v \leq 1$.

For an implementation of the solution we need to identify the Lévy system $k(y, v)$. Define by

$$J(t) = \int_0^t be^{-rs} d\gamma(cs).$$

From the Laplace transform of $J(t)$ we have

$$\begin{aligned} E\left[\exp\left(-\lambda J(t)\right)\right] &= \exp\left(\int_0^t \int_0^\infty \left(e^{-\lambda be^{-rs}x} - 1\right) \frac{dx}{x} e^{-x} c ds\right) \\ &= \exp\left(\int_0^t \int_0^\infty \left(e^{-\lambda y} - 1\right) c \exp\left(-\frac{y}{be^{-rs}}\right) \frac{1}{y} dy ds\right) \end{aligned}$$

It follows that

$$k(y,v) = \frac{c_p}{y}\exp\left(-\frac{y}{b_p v}\right)\mathbf{1}_{y>0} + \frac{c_n}{|y|}\exp\left(-\frac{|y|}{b_n v}\right)\mathbf{1}_{y<0}. \tag{22}$$

We now have all the details needed to implement the valuation through time of claims written on a large and possibly infinite maturity for a dvg driven stock price.

4 Bid and ask prices for dvg driven stock prices using probability distortions based on quadratic variation

The partial integro-differential equation is transformed into a nonlinear partial integro-differential equation to construct bid and ask prices as G-expectations. The first transformation we employ uses probability transformations on introducing a quadratic variation based probability introduced in Eberlein, Madan, Pistorius, Schoutens and Yor (2012). Specifically we rewrite the equation (20) as

$$rvH_v = \int_{-\infty}^{\infty} \frac{(H(X+y,v) - H(X,v))\int_{-\infty}^{\infty} y^2 k(y,v)dy}{y^2} dF_{QV}(y) \tag{23}$$

where

$$F_{QV}(a) = \frac{1}{\int_{-\infty}^{\infty} y^2 k(y,v)dy}\int_{-\infty}^{a} y^2 k(y,v)dy. \tag{24}$$

For the specific case considered here we have

$$\begin{aligned} F_{QV}(a) \quad = \quad & \frac{c_n(b_n v)^2}{c_p(b_p v)^2 + c_n(b_n v)^2}\left(\exp\left(-\frac{|a|}{b_n v}\right) + \left(\frac{|a|}{b_n v}\right)\exp\left(-\frac{|a|}{b_n v}\right)\right)\mathbf{1}_{a<0} + \\ & \frac{c_n(b_n v)^2}{c_p(b_p v)^2 + c_n(b_n v)^2}\mathbf{1}_{a\geq 0} + \\ & \frac{c_p(b_p v)^2}{c_p(b_p v)^2 + c_n(b_n v)^2} \times \\ & \left(1 - \exp\left(-\frac{a}{b_p v}\right) - \left(\frac{a}{b_p v}\right)\exp\left(-\frac{a}{b_p v}\right)\right)\mathbf{1}_{a>0}. \end{aligned} \tag{25}$$

We next employ the probability distortion *minmaxvar* of Cherny and Madan (2009) where

$$\Psi^\gamma(u) = 1 - (1 - u^{\frac{1}{1+\gamma}})^{1+\gamma}.$$

The nonlinear G-expectation for the bid price is then given by the solution of the distorted partial integro-differential equation

$$rvH_v = \int_{-\infty}^{\infty} \frac{(H(X+y,v) - H(X,v))\int_{-\infty}^{\infty} y^2 k(y,v)dy}{y^2} d\Psi^\gamma(F_{QV}(y)) \tag{26}$$

The ask price is computed as the negative of the bid price of the negative cash flow.

4.1 Properties of the linear expectation equation for the stock price

For the specific context of the stock price the function F is the identity function. In this case the solution of the linear expectation equation can be independently verified. Firstly one may solve explicitly for $H(X, v)$ as follows. The conditional law of $X(\infty)$ given $X(t) = X$ is that of

$$X + e^{-rt}Y = X + vY$$

where Y is an independent random variable with the same law as

$$X(\infty) = \int_0^\infty b_p e^{-ru} d\gamma_p(c_p u) - \int_0^\infty b_n e^{-ru} d\gamma_n(c_n u).$$

It follows that

$$H(X, v) = \exp(X + \omega(\infty)) \phi_Y(-iv)$$

From the characteristic function for Y we have that

$$\phi_Y(-iv) = \exp\left(\frac{c_p}{r} dilog(b_p v) + \frac{c_n}{r} dilog(-b_n v)\right).$$

It follows that

$$H(X, v) = \exp(X + \omega(\infty)) \exp\left(\frac{c_p}{r} dilog(b_p v) + \frac{c_n}{r} dilog(-b_n v)\right). \tag{27}$$

Given this solution we now verify directly the differential equation, with a view to adjusting it to work numerically with a grid on the stock price as opposed to a space grid in X. The stock price process is driftless while the process $X(t)$ can have a substantial drift.

Consider then the integral in the equation (20). Explicitly for the claim paying the stock we have that

$$\int_{-\infty}^\infty (H(X + y, v) - H(X, v)) k(y, v) dy = H(X, v) \int_{-\infty}^\infty (e^y - 1) k(y, v) dy.$$

For the left hand side differentiating (27) we have that

$$H_v(X, v) = H(X, v) \left(\frac{c_p b_p}{r} dilog'(b_p v) - \frac{b_n c_n}{r} dilog'(-b_n v)\right).$$

Now we use the fact that

$$dilog'(a) = -\frac{\ln(1-a)}{a}$$

we write

$$H_v(X,v) = H(X,v)\left(-\frac{c_p \ln(1-b_p v)}{rv} - \frac{c_n \ln(1+b_n v)}{rv}\right)$$

or that

$$rvH_v = H(X,v)\left(-c_p \ln(1-b_p v) - c_n \ln(1+b_n v)\right)$$

The differential equation is then verified on recalling the relationship between the logarithm of the Laplace transform for the gamma process and its Lévy measure whereby we have that

$$\int_0^\infty \left(e^{-\lambda x} - 1\right)\frac{\gamma}{x}e^{-cx}dx = -\gamma \ln\left(1 + \frac{\lambda}{c}\right).$$

The use of this result with $\lambda = -1, 1$ $\gamma = c_p, c_n$ and $c = 1/(b_p v)$, $1/(b_n v)$ for the integrals on the positive and negative sides respectively yields

$$\int_{-\infty}^\infty \left(e^y - 1\right)k(y,v)dy = -c_p \ln(1-b_p v) - c_n \ln(1+b_n v).$$

As commented earlier, for numerical solutions it is preferable to have a stationary grid for the space variable and this is expected for the discounted stock price. We are therefore led to write

$$\begin{aligned} H(X,v) &= \exp\left(X + \omega\left(t\right) + \omega(\infty) - \omega(t)\right)\phi_Y(-iv) \\ t &= -\frac{\ln v}{r}. \end{aligned}$$

Further observing that

$$M(t) = \exp\left(X(t) + \omega(t)\right)$$

define

$$G(M(v),v) = M(v)\exp\left(\omega(\infty) - \omega(-\frac{\ln v}{r})\right)\phi_Y(-iv).$$

Dropping for notational convenience the dependence of M on v we write that

$$\int_{-\infty}^\infty (G(Me^y,v) - G(M,v))k(y,v)dy = G(M,v)\int_{-\infty}^\infty \left(e^y - 1\right)k(y,v)dy.$$

Also we have that

$$\begin{aligned} G_v &= G(M,v)\left(\frac{\partial \ln \phi_Y(-iv)}{\partial v} + \frac{1}{rv}\omega'\left(-\frac{\ln v}{r}\right)\right) \\ &= G(M,v)\left(-\frac{c_p \ln(1-b_p v)}{rv} - \frac{c_n \ln(1+b_n v)}{rv} + \frac{1}{rv}\omega'\left(-\frac{\ln v}{r}\right)\right) \end{aligned}$$

It follows that $G(M,v)$ satisfies the differential equation

$$rvG_v = \int_{-\infty}^{\infty} (G(Me^y, v) - G(M,v))k(y,v)dy + G(M,v)\omega' \left(-\frac{\ln v}{r}\right) \tag{28}$$

One may therefore work on a fixed stock grid centered around unity with the differential equation (28) on applying the desired distortions to the Lévy system $k(y,v)$. The function $\omega'(t)$ may be precomputed.

The discretized update for the conditional expectation of $M(\infty)$ is now

$$G(M,v+h) = G(M,v) + \frac{h}{rv} \times \left(\int_{-\infty}^{\infty} (G(Me^y, v) - G(M,v))k(y,v)dy + G(M,v)\omega' \left(-\frac{\ln v}{r}\right) \right). \tag{29}$$

However, it will be useful to incorporate the analytical solution to (28) into the numerical scheme. Note that when $X(t) = X$ we have

$$M(t) = \exp\left(X + \omega(t)\right)$$

but as $M(t)$ is a uniformly integrable martingale we must have

$$\begin{aligned} E_t\left[\exp\left(X(\infty) + \omega(\infty)\right)\right] &= M(t) \\ &= \exp\left(X + \omega(t)\right) \end{aligned}$$

But this conditional expectation is

$$\exp\left(X + \omega(\infty)\right)\phi_Y(-iv).$$

Hence one has the implication that

$$\phi_Y(-iv) = \exp\left(\omega(-\frac{\ln v}{r}) - \omega(\infty)\right)$$

This implication may be independently verified on observing that as

$$\ln\left(\phi_Y(-iv)\right) = \frac{c_p}{r} dilog\left(b_p v\right) + \frac{c_n}{r} dilog\left(-b_n v\right)$$

it must be the case that this value coincides with

$$\omega(-\frac{\ln v}{r}) - \omega(\infty).$$

From the characteristic function of $X(t)$ we see that

$$\begin{aligned} \omega(t) &= \frac{c_p}{r}\left(dilog\left(b_p e^{-rt}\right) - dilog\left(b_p\right)\right) \\ &\quad + \frac{c_n}{r}\left(dilog\left(-b_n e^{-rt}\right) - dilog\left(b_n\right)\right) \end{aligned}$$

From which we see that in fact

$$\omega\left(-\frac{\ln v}{r}\right) - \omega(\infty) = \ln\left(\phi_Y(-iv)\right).$$

It follows that

$$M \exp\left(\omega(\infty) - \omega\left(-\frac{\ln v}{r}\right)\right) \phi_Y(-iv) = M$$

and the solution of the differential equation

$$rvG_v = \int_{-\infty}^{\infty} \left(G(Me^y, v) - G(M, v)\right) k(y, v) dy + G(M, v)\omega'\left(-\frac{\ln v}{r}\right) \tag{30}$$

is in fact

$$G(M, v) = M.$$

4.2 Implementation details

The pricing is implemented for risk neutral parameter values for the S&P 500 index taken at their median values as reported in Madan and Yor (2012). These are

$$\begin{aligned} r &= .02966 \\ b_p &= 0.0145 \\ c_p &= 48.4215 \\ b_n &= 0.5707 \\ c_n &= 0.3493 \end{aligned}$$

The differential equation solved for the bid price is

$$rvG_v = \int_{-\infty}^{\infty} \frac{\left(G(Me^y, v) - G(M, v)\right) \int_{-\infty}^{\infty} y^2 k(y, v) dy}{y^2} d\Psi^{\gamma}\left(F_{QV}(y)\right) + G(M, v)\omega'\left(-\frac{\ln v}{r}\right). \tag{31}$$

In the absence of a distortion the equation has the solution $G(M, v) = M$. In the computations we set ω' to $\widehat{\omega}'$ that forces the expectation equation (30) to solve out at the identity function.

Hence we set

$$\widehat{\omega}'(v) = -\frac{\int_{-\infty}^{\infty}(G(Me^y, v) - G(M, v))k(y, v)dy}{G(M, v)}$$

in the solution of the expectation equation (30). This value of $\widehat{\omega}'$ is then used in the bid and ask equations. It was checked that the values for $\widehat{\omega}'$ and ω' were close.

For this parameter setting and with the minmaxvar stress level set at 10 basis points the bid and ask prices were solved for as a function of the spot on the initial date. The result is presented in Figure 1.

We also present in Figure 2 a graph of the bid and ask prices as a function of calendar time for different levels of the initial spot. The prices converge at infinity to the expected value.

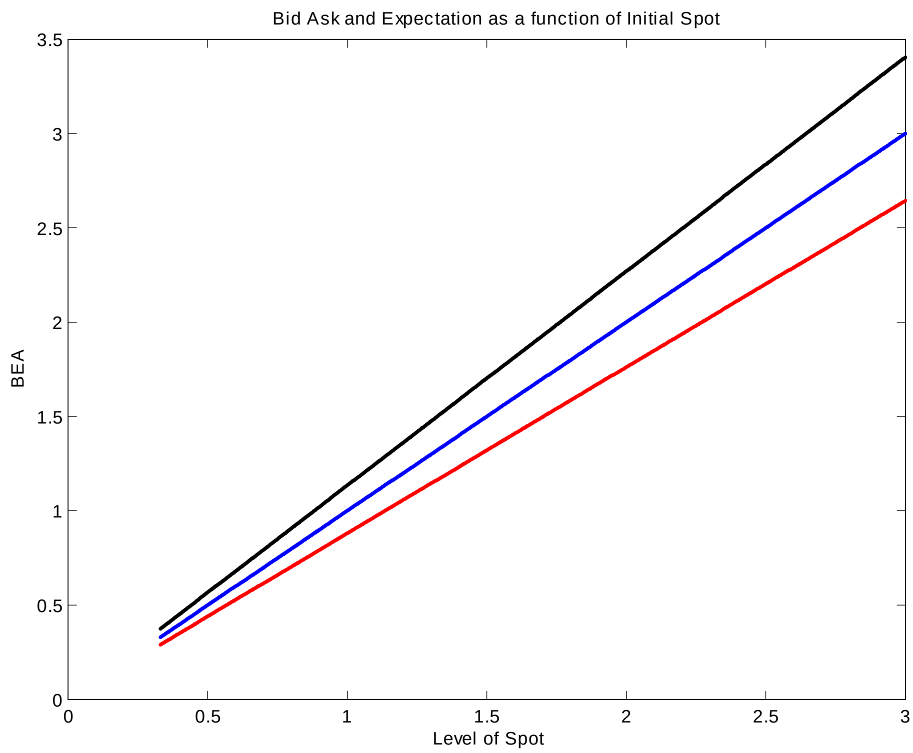

Figure 1: Bid, ask and expectation as a function of the spot price at time zero.

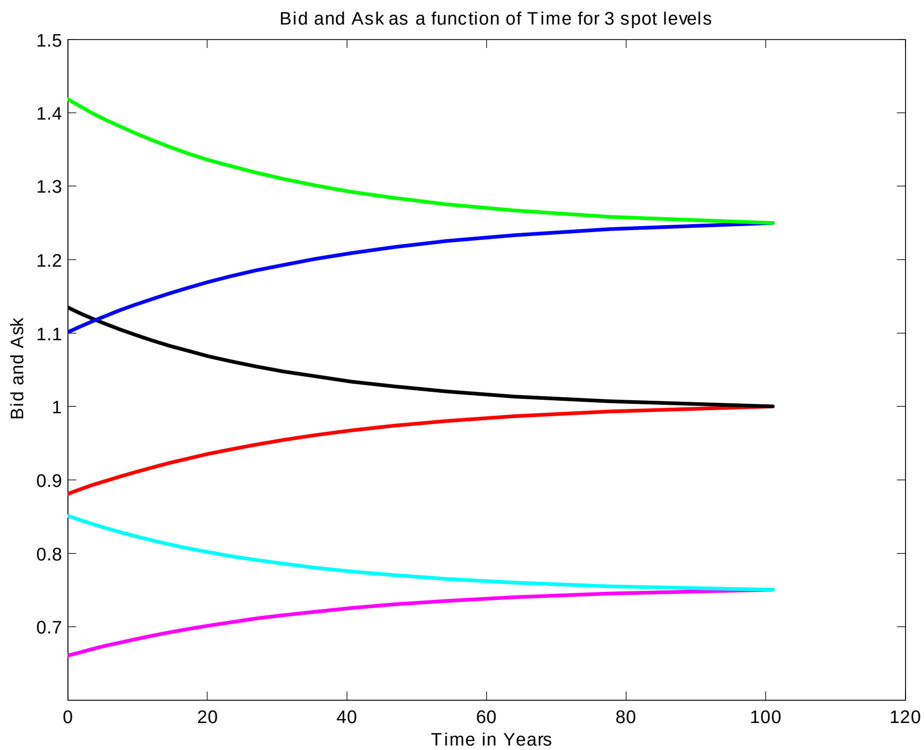

Figure 2: Bid and ask prices as functions of time for three different spot levels using measure distortions. The lower curves in magenta and cyan are the bid and ask for the spot level of 0.75. Red and black are the bid and ask for the level 1.0 and blue and green are for level 1.25.

5 Bid and ask for dvg driven stock using measure distortions

The first step in applying measure distortions is that of choosing specific functional forms for the measure distortions Γ_+, Γ_-. Recognizing that Γ_+ lies above the identity and Γ_- lies below we consider functional forms for the positive gap

$$\begin{aligned} G_+(x) &= \Gamma_+(x) - x \\ G_-(x) &= x - \Gamma_-(x) \end{aligned}$$

Both these functions are concave and positive. If we suppose that for large x associated with large tail measures and therefore events nearer to zero, there need be no reweighting then one has Γ'_+ falling to unity at large x while Γ'_- rises to unity. As a result G_+, G_- are increasing concave functions that are eventually constant. We may scale by the final constant and model them to be multiples of increasing concave functions that are finally unity. We then write

$$\begin{aligned} G_+(x) &= \alpha K_+(x) \\ G_-(x) &= \beta K_-(x) \end{aligned}$$

where K_+, K_- are unity at infinity.

Now consider a generic candidate for such a function, say $K(x)$. Suppose the concavity coefficient defined by

$$-\frac{K''}{K'}$$

is bounded below by a constant $c > 0$. Define

$$\Psi(y) = K\left(-\frac{\ln(1-y)}{c}\right), \quad 0 \le y \le 1.$$

The function Ψ is zero at zero, unity at unity, and increasing in its domain. Furthermore we have

$$\begin{aligned} \Psi'(y) &= K'\left(-\frac{\ln(1-y)}{c}\right) \times \left(\frac{1}{c(1-y)}\right) \\ \Psi''(y) &= K''\left(-\frac{\ln(1-y)}{c}\right) \times \left(\frac{1}{c^2(1-y)^2}\right) \\ &\quad + K'\left(-\frac{\ln(1-y)}{c}\right) \times \frac{1}{c(1-y)^2} \end{aligned}$$

and $\Psi'' \le 0$ just if

$$\frac{K''}{c} + K' \le 0$$

or

$$-\frac{K''}{K'} \ge c.$$

With a lower bound on the concavity coefficient we have that

$$K(x) = \Psi\left(1 - e^{-cx}\right)$$

and Ψ is a probability distortion.

Hence we take as models for specific measure distortions

$$\begin{aligned}\Gamma_+(x) &= x + \alpha\Psi_+\left(1 - e^{-cx}\right)\\ \Gamma_-(x) &= x - \frac{\beta}{c}\Psi_-(1 - e^{-cx})\end{aligned}$$

for any probability distortions Ψ_+, Ψ_-. The distortions maxvar, $\Phi^{\gamma}_{\max}(u)$ and minvar, $\Phi^{\gamma}_{\min}(u)$ are defined by

$$\begin{aligned}\Phi^{\gamma}_{\max}(u) &= u^{\frac{1}{1+\gamma}}\\ \Phi^{\gamma}_{\min}(u) &= 1 - (1-u)^{1+\gamma}.\end{aligned}$$

If one takes maxvar for Ψ_+ to get an infinite reweighting of large losses and minvar for Ψ_- we have the specific formulation

$$\begin{aligned}\Gamma_+(x) &= x + \alpha\left(1 - e^{-cx}\right)^{\frac{1}{1+\gamma_+}}\\ \Gamma_-(x) &= x - \frac{\beta}{c}(1 - e^{-c(1+\gamma_-)x})\end{aligned}$$

In the calculations reported we set $\gamma_- = 0$ and employed a four parameter specification for the measure distortion with the parameters α, β, c and $\gamma_+ = \gamma$. The maximum downward discounting of gains is $\Gamma'_-(0) = 1 - \beta$.

5.1 Measure distortion results for the dvg stock price

Distorting the integral in equation (30) we get according to equations (8)-(10) in the case of the dvg stock price

$$\begin{aligned}rvG_v = &\int_{-\infty}^{0} xd\left(\Gamma_+\left(\int_{(G(Me^y,v)-G(M,v)\leq x)} k(y,v)dy\right)\right) -\\ &\int_{0}^{\infty} xd\left(\Gamma_-\left(\int_{(G(Me^y,v)-G(M,v)>x)} k(y,v)dy\right)\right)\\ &+ G(M,v)\omega'\left(-\frac{\ln v}{r}\right).\end{aligned}$$

The results shown are for

$$\begin{aligned}\alpha &= 0.01\\ \beta &= 0.05\\ c &= 1\\ \gamma &= 0.0010\end{aligned}$$

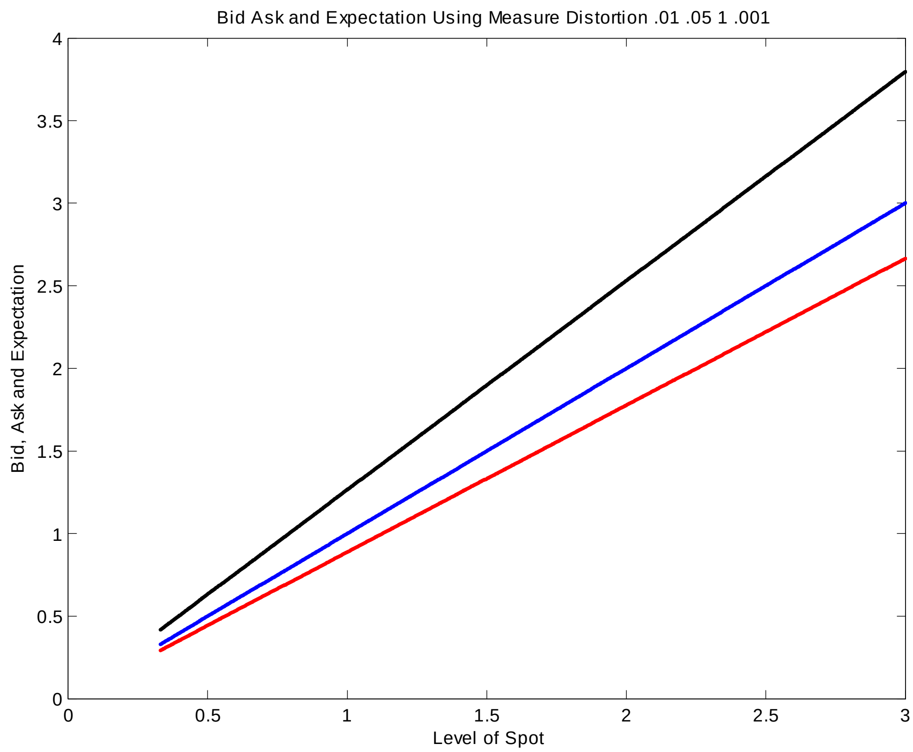

Figure 3: Bid, ask and expected values as a function of the spot at the initial time using measure distortions.

Figure 3 presents the bid, ask and expectation as a function of the initial spot. We also present the bid and ask as functions of time for three different spot levels in Figure 4

6 Two price valuation of insurance loss processes

This section applies measure distortions to the two price valuation of insurance losses. Let $L(t)$ be the process for cumulated losses. A discounted expected value may be computed as

$$E\left[\int_0^\infty e^{-rs} dL(s)\right]$$

where $L(t)$ is for example a compound Poisson process with arrival rate λ and loss sizes that are *i.i.d.* gamma distributed with scale and shape parameters ζ and κ respectively. Consider the value process in time zero dollars for these

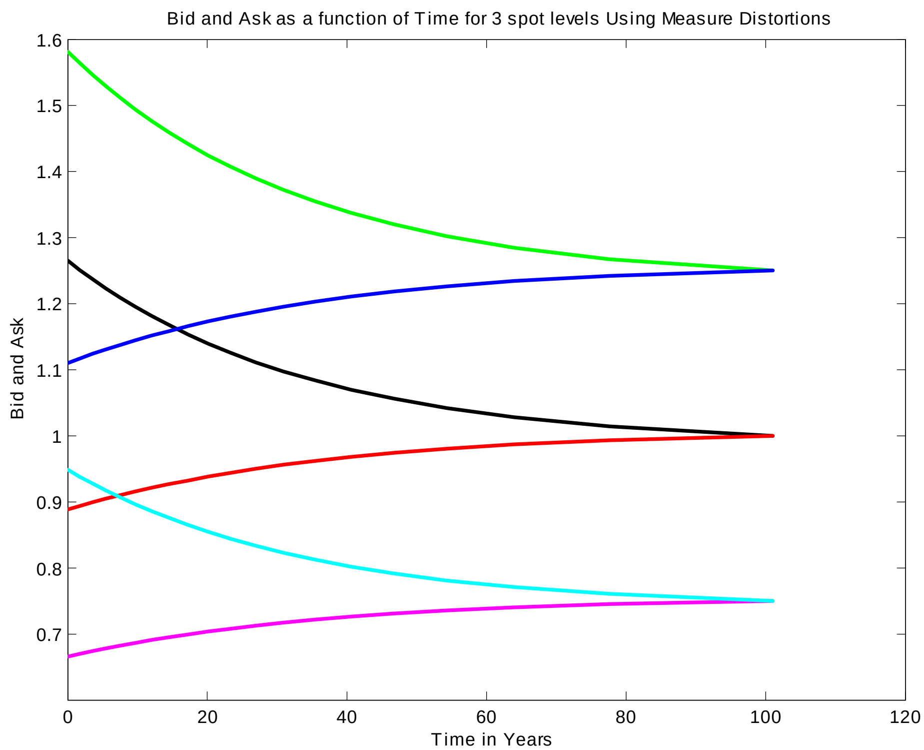

Figure 4: Bid and ask as functions of time for three different spot levels using measure distortions. The lower curves in magenta and cyan are the bid and ask for the spot level of 0.75. Red and black are the bid and ask for the level 1.0 and blue and green are for level 1.25.

losses,

$$V(t) = E_t \left[\int_0^\infty e^{-rs} dL(s) \right], \tag{32}$$

where $E_t[.]$ denotes the operator for expectation conditional on information at time t.

Let $X(t)$ be the level of discounted losses to date or

$$X(t) = \int_0^t e^{-rs} dL(s).$$

We then write

$$\begin{aligned} \int_0^\infty e^{-rs} dL(s) &= X(t) + e^{-rt} \int_t^\infty e^{-r(s-t)} dL(s) \\ &\stackrel{(d)}{=} X(t) + e^{-rt} Y \end{aligned}$$

where Y is an independent copy of the random variable

$$\int_0^\infty e^{-rs} dL(s).$$

It follows that the conditional expectation is a martingale of the form

$$H(X(t), e^{-rt}).$$

Using the time transformation $v = e^{-rt}$ the martingale condition for H once again yields that

$$rvH_v = \int_0^\infty (H(X+w,v) - H(X,v)) k(w,v) dw$$

where $k(w,v)$ is related to the Lévy system for $X(t)$.

We may derive this Lévy system from the characteristic function for $X(t)$. The characteristic function is developed as follows

$$\begin{aligned} E\left[\exp\left(iuX(t)\right)\right] &= E\left[\exp\left(iu \int_0^t e^{-rs} dL(s)\right)\right] \\ &= \exp\left(\int_0^t \int_0^\infty \left(e^{iue^{-rs}x} - 1\right) \frac{\lambda}{\Gamma(\kappa)} \zeta^\kappa x^{\kappa-1} e^{-\zeta x} dx ds\right) \\ &= \exp\left(\int_0^t \int_0^\infty \left(e^{iuw} - 1\right) \frac{\lambda}{\Gamma(\kappa)} \left(\frac{\zeta}{e^{-rs}}\right)^\kappa w^{\kappa-1} \exp\left(-\frac{\zeta}{e^{-rs}} w\right) dw ds\right) \end{aligned}$$

It follows that the Lévy system for $X(t)$ is

$$k(w,t) = \frac{\lambda}{\Gamma(\kappa)} \left(\frac{\zeta}{e^{-rt}}\right)^\kappa w^{\kappa-1} \exp\left(-\frac{\zeta}{e^{-rt}} w\right)$$

6.1 Remarks on the expectation equation

We already know that

$$H(X,v) = X + vC$$

where $C = E_t[Y] = E[Y]$. Hence

$$\begin{aligned}\int_0^\infty (H(X+w,v) - H(X,v))k(w,v)dw &= \int_0^\infty wk(w,v)dw \\ &= \frac{\lambda\kappa v}{\zeta}\end{aligned}$$

since $k(w,v)$ is the density of a gamma variable with scale and shape ζ/v and κ respectively.

Consequently

$$\frac{1}{rv}\int_0^\infty (H(X+w,v) - H(X,v))k(w,v)dw = \frac{\lambda\kappa}{r\zeta}$$

On the other hand

$$H_v = C$$

We now develop the characteristic function for Y. The characteristic function of Y is

$$\begin{aligned}E\left[\exp\left(iuY\right)\right] &= E\left[\exp\left(iu\int_0^\infty e^{-rs}dL(s)\right)\right] \\ &= \exp\left(\int_0^\infty ds\int_0^\infty \left(e^{iue^{-rs}x} - 1\right)\lambda\frac{\zeta^\kappa x^{\kappa-1}e^{-\zeta x}}{\Gamma(\kappa)}dx\right) \\ &= \exp\left(\int_0^\infty ds\int_0^\infty \left(e^{iuw} - 1\right)\frac{\lambda\zeta^\kappa}{\Gamma(\kappa)}(we^{rs})^{\kappa-1}\exp\left(-\zeta we^{rs}\right)e^{rs}dw\right) \\ &= \exp\left(\int_0^1 dv\int_0^\infty \left(e^{iuw} - 1\right)\frac{\lambda\zeta^\kappa}{\Gamma(\kappa)rv^2}\left(\frac{w}{v}\right)^{\kappa-1}\exp\left(-\frac{\zeta w}{v}\right)dw\right) \\ &= \exp\left(\int_0^1 dv\int_0^\infty \left(e^{iuw} - 1\right)\frac{k(w,v)}{rv}dw\right)\end{aligned}$$

We may determine C from the derivative of the characteristic function of Y.

$$\phi'_Y(u) = \exp\left(\int_0^1 dv\int_0^\infty \left(e^{iuw} - 1\right)\frac{k(w,v)dw}{rv}\right)\int_0^1 dv\int_0^\infty \frac{1}{rv}iwe^{iuw}k(w,v)dw$$

Evaluating at $u = 0$ and multiplying by $-i$ yields

$$\begin{aligned}C &= -i\phi'_Y(0) = \int_0^1 dv\int_0^\infty \frac{1}{rv}wk(w,v)dw \\ &= \frac{\lambda\kappa}{r\zeta}\end{aligned}$$

Hence the differential equation holds.

6.2 Implementing insurance loss valuation

We consider an arrival rate $\lambda = 10$ with a gamma distribution of mean 5 and variance 10. Therefore we have $\zeta = .5$, $\kappa = 2.5$ and $\lambda = 10$. We take the interest rate at $r = .02$. The mean of the final discounted loss is

$$\frac{\lambda\kappa}{r\zeta} = 2500$$

The process for X starts at zero and finishes at a mean of 2500.

The differential equation is

$$H_v = \frac{1}{rv}\int_0^\infty (H(X+y,v) - H(X,v))k(y,v)dy$$

with

$$k(y,v) = \frac{\lambda}{\Gamma(\kappa)}\left(\frac{\zeta}{v}\right)^\kappa y^{\kappa-1}\exp\left(-\frac{\zeta}{v}y\right)$$

We fix a grid in X from 0 to 100 measured in thousands for which we take $\zeta = 500$. We take the grid in X to be in the range 0.25 to 100 in the step size of 0.25.

We only have positive outcomes for the cumulated discounted loss process. The bid and ask prices are then respectively the solutions to (see formula (10) for nonnegative functions $v(y)$)

$$rvH_v = -\int_0^\infty xd\Gamma_-\left(\mu\left(\chi > x\right)\right)$$

and

$$rvH_v = -\int_0^\infty xd\Gamma_+\left(\mu\left(\chi > x\right)\right),$$

where $\chi(y) = H(X+y,v) - H(X,v) = v(y)$ in formula (10). The measure $\mu(dy)$ is $k(y,v)dy$.

We have the same equation but we use Γ_- for the bid and Γ_+ for the ask. The measure distortion parameters used were $\alpha = .1$, $\beta = .2$, $c = 1$ and $\gamma = .02$.

We present in Figures 5 and 6 the graphs for the bid, ask and expectation as functions of the initial loss level and as functions of time for three loss levels respectively.

7 Remarks on the design of measure distortions

There are four parameters in the proposed measure distortion Γ_+, Γ_- and they are α, β, c and γ. The parameter c may be calibrated by a cutoff on what are viewed as rare events. If the exponential of $-cx$ is below $1/2$ then $1-\exp(-cx) > 1/2$ and these are the likely events. Defining

$$x_* = \frac{-\ln(1/2)}{c}$$

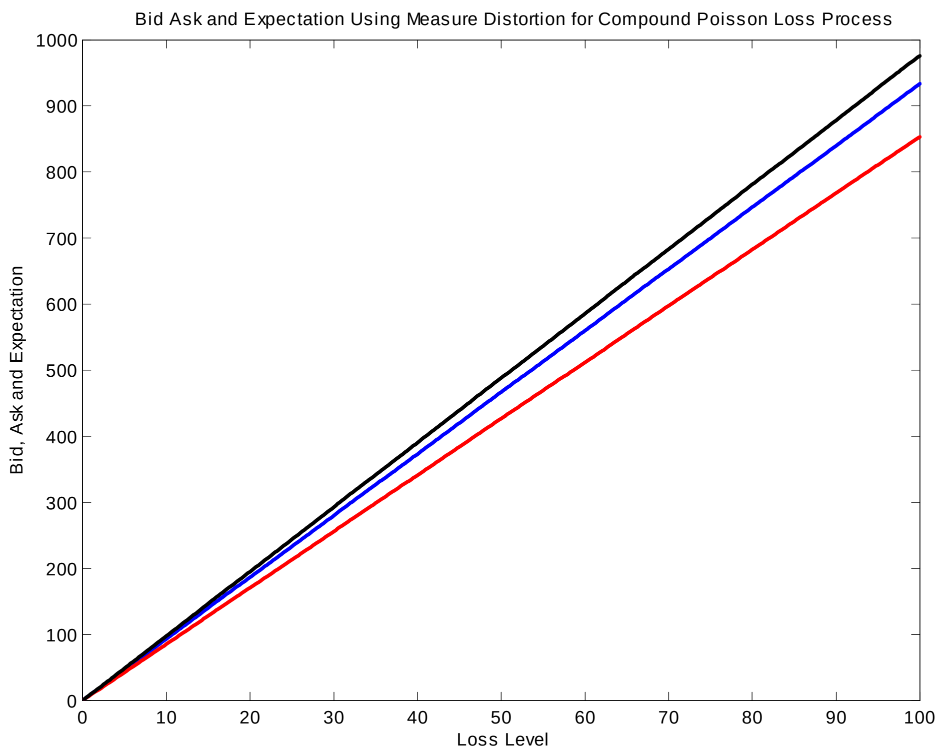

Figure 5: Bid, ask and expectation as functions of initial loss level with the lower line being the bid and the upper line the ask.

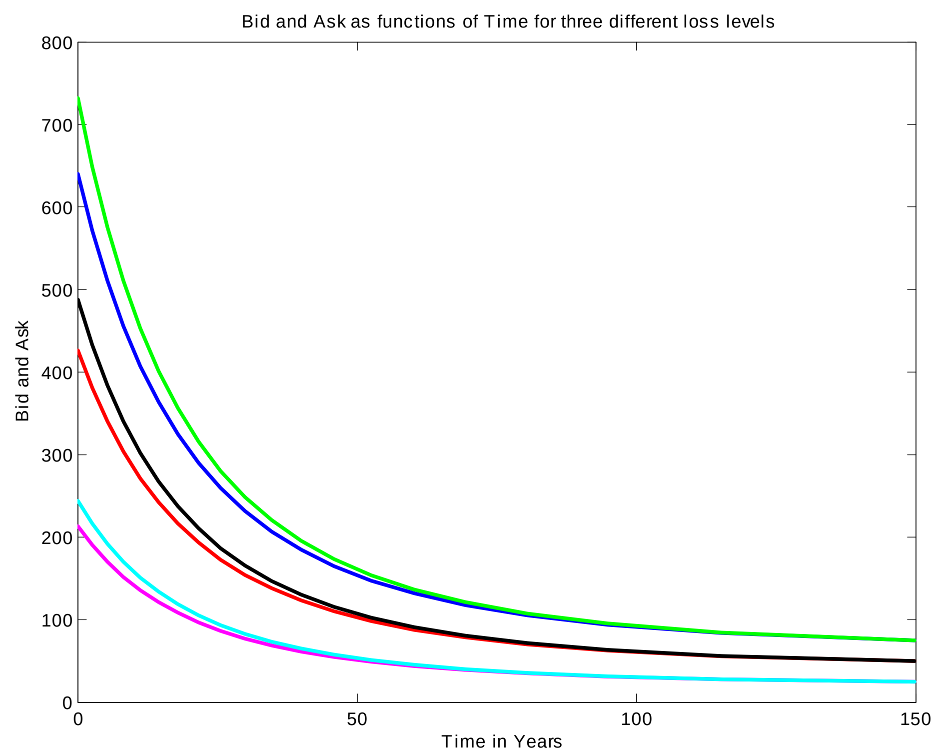

Figure 6: Bid and ask as functions of time for three different loss levels. The lower curves in magenta and cyan are the bid and ask for the loss level of 25. Red and black are the bid and ask for the level 50 and blue and green are for level 75.

we have arrival rates below x_* constituting the rare events. Hence for $c = -\ln(1/2)$ arrival rates above one per year are the normal events while arrival rates below one per year are the rare ones. If arrival rates below 2 per year are to be the rare ones then $c = -\ln(1/2)/2 = 0.3466$ and if rare is viewed as one every two years then $c = 1.3863$.

The parameter β sets the discount on gains. For $\beta = 0$ there is no gain discount and Γ_- is the identity function. The highest gain discount is unity. The gain discount should be set below unity.

Once β and c are set then the choice of η in

$$\alpha = \frac{\beta}{c}\eta$$

sets the parameter α. The choice of $\eta = 1$ provided a balanced treatment of gains and losses as the maximum penalty in the gap functions G_+ and G_- are then equal. The parameter η is then a balance parameter

The parameter γ is a stress parameter and controls the speed with which losses are reweighted upwards. This is a parameter familiar from the uses of the probability distortions *maxvar* or *minmaxvar*. The next section provides some parameter sensitivities.

8 Inhomogeneous compound Poisson losses

Consider an inhomogeneous arrival rate $\lambda(t)$ for losses, a general discount curve $D(t)$ and gamma distributed loss sizes. For the loss process $L(t)$ the linear finite expectation valuation is given by

$$V(t) = E_t\left[\int_0^\infty D(s)dL(s)\right] \tag{33}$$

Now for any t we may write

$$V(t) = \int_0^t D(s)dL(s) + E_t\left[\int_t^\infty D(s)dL(s)\right]$$

Define

$$C(t) = \int_t^\infty D(s)\int_0^\infty \lambda(s)\frac{\zeta^\kappa}{\Gamma(\kappa)}x^\kappa e^{-\zeta x}dxds$$

and observe that $N(t)$ defined as

$$N(t) = \int_0^t D(s)dL(s) + C(t)$$

is a martingale. In fact

$$dN(t) = D(t)dL(t) - D(t)\int_0^\infty \lambda(t)\frac{\zeta^\kappa}{\Gamma(\kappa)}x^\kappa e^{-\zeta x}dxdt$$

and as the compensator of $dL(t)$ is

$$\lambda(t)\frac{\zeta^{\kappa}}{\Gamma(\kappa)}x^{\kappa-1}e^{-\zeta x}dxdt$$

we have a martingale.

In general in the current context we have for $X(t) = \int_0^t D(s)dL(s)$ that

$$V(t) = H(X(t), t)$$

where in fact the function H takes the specific form

$$H(X(t), t) = X(t) + C(t).$$

Apply Ito's lemma to the function H and noting that it is a martingale we deduce that

$$H_t + \int_0^{\infty} (H(X(t) + y, t) - H(X(t), t))k(y, t)dy = 0$$

where again $k(y, t)$ is the Lévy system for $X(t)$. Equivalently in terms of the compensator for $dL(t)$ we may write

$$H_t = -\int_0^{\infty} (H(X(t) + D(t)x, t) - H(X(t), t))\lambda(t)\frac{\zeta^{\kappa}}{\Gamma(\kappa)}x^{\kappa-1}e^{-\zeta x}dx.$$

Now substituting the specific form of the function H yields that

$$C_t = -\int_0^{\infty} D(t)x\lambda(t)\frac{\zeta^{\kappa}}{\Gamma(\kappa)}x^{\kappa-1}e^{-\zeta x}dx$$

or that

$$C(t) = \int_t^{\infty}\int_0^{\infty} D(s)x\lambda(s)\frac{\zeta^{\kappa}}{\Gamma(\kappa)}x^{\kappa-1}e^{-\zeta x}dxds.$$

For the nonlinear measure distorted result we write

$$\mathcal{E}_t\left[\int_0^{\infty} D(s)dL(s)\right] = \int_0^t D(s)dL(s) + \widetilde{C}(t)$$

$$\widetilde{C}(t) = \int_t^{\infty} D(s)\int_0^{\infty} \Gamma_+\left(\int_x^{\infty} \lambda(t)\frac{\zeta^{\kappa}}{\Gamma(\kappa)}y^{\kappa-1}e^{-\zeta y}dy\right)dxds \qquad (34)$$

The computation of $\widetilde{C}(t)$ as the ask price valuation is what we implement for the inhomogeneous case as a conjectured solution. For the bid we replace Γ_+ by Γ_-.

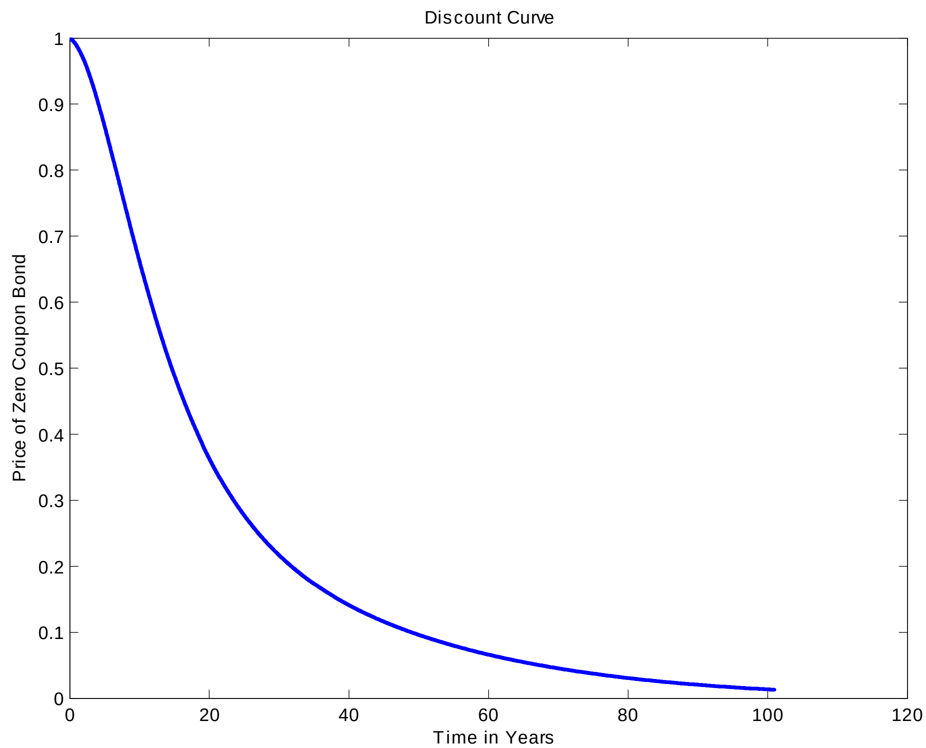

Figure 7: Discount curve used in inhomogeneous compound Poisson loss model.

8.1 Inhomogeneous example

We employ for the discount curve a Nelson-Siegel discount curve with yield to maturity $y(t)$ specification

$$\begin{aligned} y(t) &= a_1 + (a_2 + a_3 t)e^{-a_4 t} \\ a_1 &= .0424 \\ a_2 &= -.0367 \\ a_3 &= .0034 \\ a_4 &= .0686. \end{aligned}$$

The discount curve is graphed in Figure 7.

For the inhomogeneous arrival rate we take an exponential model with

$$\lambda(t) = \frac{k}{\tau} \exp\left(-\frac{t}{\tau}\right).$$

The parameters used were $k = 100$ and $\tau = 10$. The distribution for losses are gamma with $\zeta = \kappa = 1.2346$, consistent with a unit mean and a standard

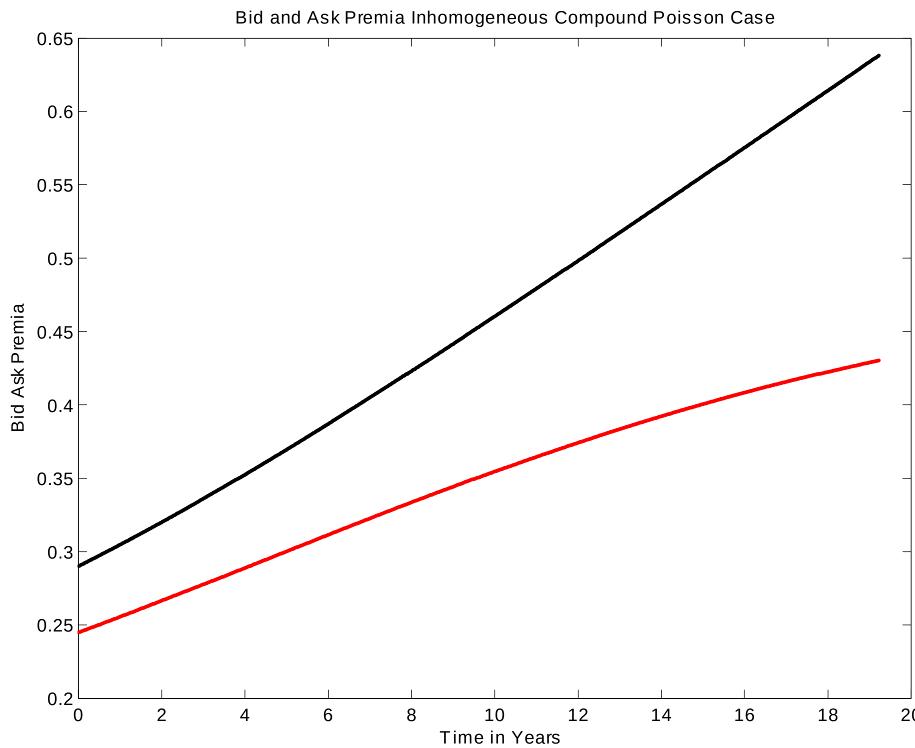

Figure 8: Bid and ask premia in basis points relative to expected values for the inhomogeneous compound Poisson case.

deviation of 0.9. For the measure distortions parameters we employ $\alpha = .7214$, $\beta = .5$, $c = .6931$ and $\gamma = 0.25$. Presented in Figure 8 are the premia of ask over expectation and the shave of bid relative to expectation in basis points, for the function $\widetilde{C}(t)$ when we use Γ_+ for the ask and Γ_- for the bid in the expression (34).

With a view to understanding the effect of various parameters we present a set of graphs of the effects of various parameters in the gamma compound Poisson case on the Γ_+, Γ_- measure distorted compensation premia over the expectation. The base parameter setting is an arrival rate $\lambda = 50$, a mean loss size of 3 with a volatility of 0.75. The parameter for balance fixes $\alpha = \beta/c$. The base case value for β is 0.25, for c it is 0.6931 and for γ it is 0.25. The parameters are then varied in turn through a range for which we compute the measure distorted instantaneous compensation. Figure 9 presents the results.

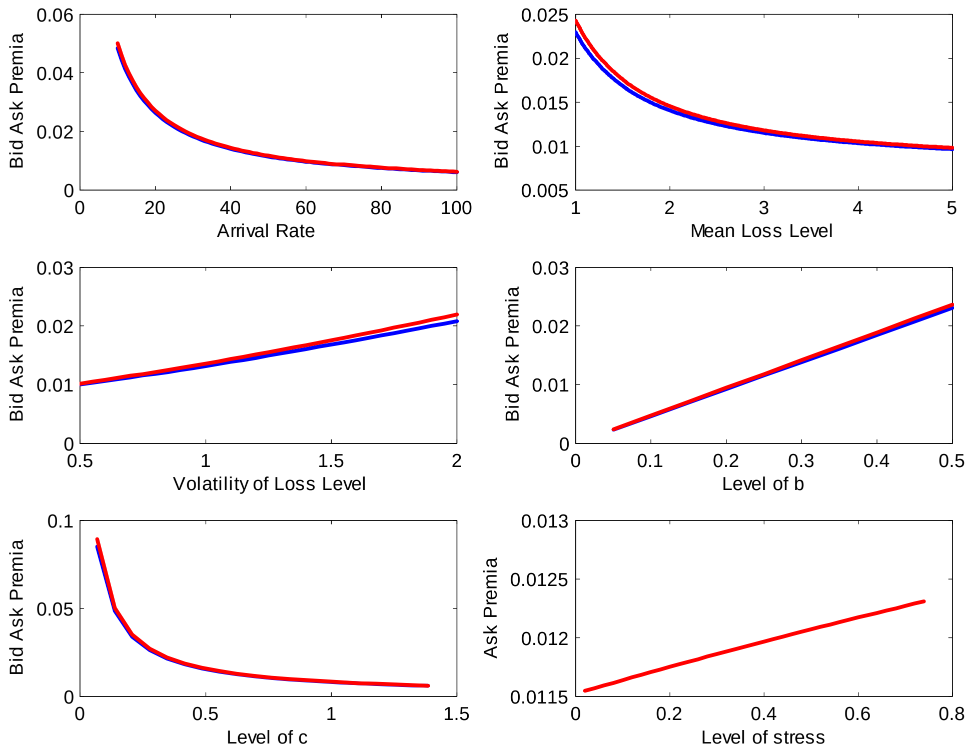

Figure 9: The effect of various parameters on the instantaneous measure distorted compensation for the gamma compound Poisson risk.

9 Conclusion

The use of probability distortions in constructing nonlinear G-expectations for bid and ask or lower and upper prices in continuous time as introduced in Eberlein, Madan, Pistorius, Schoutens and Yor (2012) is here extended to the direct use of measure distortions. Integrals with respect to a possibly infinite measure with finite measure in the two sided tails on either side of zero are distorted using concave measure distortions for losses and convex measure distortions for gains. It is shown that measure distortions can fairly generally be constructed as probability distortions applied to an exponential distribution function on the half line.

The valuation of economic activities as opposed to contracts places the problem in a context with no apparent maturity. The two price continuous time methodologies heretofore available for explicit maturities are extended to economic activities with infinite lives. This permits the construction of two prices for stock indices and the coverage of insurance liabilities in perpetuity.

The methods are illustrated with explicit computations using probability and measure distortions for an infinitely lived stock price model as developed in Madan and Yor (2012). Measure distortions are applied to infinitely lived compound Poisson insurance loss processes.

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