On Correlating Lévy Processes

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Abstract

A relatively simple approach to correlating unit period returns of Lévy processes is developed. We write the Lévy process as a time changed Brownian motion and correlate the Brownian motions. It is shown that sample correlations understate the required correlation between the Brownian motions and we show how to correct for this. Pairwise tests illustrate the adequacy of the model and the significant improvement offered over the Gaussian alternative. We therefore advocate that the correlated time change model is a simple basic alternative to dependence modeling. From the perspective of explaining portfolio returns in higher dimensions we find adequacy for long-short portfolios. The long only portfolios appear to require a more complex modeling of dependency. We leave these questions for future research.

Keywords: Dependence Modeling, Levy Copulas, Multivariate Subordination, Long-Short vs Long only Portfolios, Variance Gamma, Time Changed Brownian Motion

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Lévy processes offer a rich class of distributions for modeling financial returns and have been successfully used to capture varying levels of skewness and kurtosis seen in both the physical and risk neutral return densities. By way of examples demonstrating this flexibility we cite Eberlein (2001) and Carr, Geman, Madan and Yor (2002). The Gaussian distribution, associated with the Brownian motion process, on the other hand, can only capture the level of volatility thereby making this distribution ill suited to describing risk neutral distributions that often display, for example, high levels of skewness. For the physical distribution of daily returns, on the other hand, it is high levels of kurtosis that are important. As marginal return distributions are both non-Gaussian and well characterized by the unit time distribution of a Lévy process and since returns across assets are clearly not independent, the modeling of dependence in Lévy processes is called for. Additionally there is a need for some efficiency in this modeling particularly with respect to the analysis of risks embedded in multiasset structured products.

We note in this connection that in the risk neutral domain the departures from the Gaussian model are referred to as the implied volatility smile (Derman and Kani (1994), Dupire (1994), Rubinstein (1994)) and they have resulted in the development of the local volatility model as a resolution. In the local volatility model the local motion of log prices remains a Brownian motion but the volatility now continuously responds to movements in the asset price and the passage of time. The additional flexibility accessed is capable of simultaneously delivering a large class of risk neutral distributions at a continuum of maturities as explained in some generality by Gyöngy (1996). This model is now widely used in the analysis of risks in equity structured products.

The local volatility model has some well known problems. Among them is the observation of declining skews in the forward implied volatility curves of local volatility models (Gatheral (2006)). A resolution was provided by the generalization of local volatility models to local Lévy models (Carr, Geman, Madan and Yor (2005)). In a local Lévy model the local motion of log prices comes from a skewed Lévy process running at a speed continuously responding to the level of the asset price and calendar time. The presence of a skewed local motion helps keep forward implied volatility skews from collapsing.

Once local volatility or local Lévy models have been fit to the option surfaces of a number of underliers the marginal option surfaces of all underliers are then calibrated. For structured products with payoffs depending on multiple underliers it becomes necessary to construct correlated path spaces. In the local volatility context this is typically done by correlating the Brownian motions driving the various underlying assets. For the local Lévy model there is again the need to correlate the Lévy processes.

Correlating Lévy processes however, is relatively a more difficult task. Unlike Brownian motion which is always active, pure jump Lévy processes, even when they are infinitely active (with an infinite number of jumps or moves in any interval) are actually silent with no moves at any finite set of prespecified times. For two independent Lévy processes, when one moves the other has no move and so it becomes difficult to correlate action with no action. An approach that has been taken is to introduce a common time change to force all assets to move at precisely the same instant and we refer to Luciano and Schoutens (2006), Semeraro (2006), and Luciano and Semeraro (2007). Yet another approach is to consider a linear mixture of independent Lévy processes as studied in Ballotta (2008), Madan and Yen (2008), and Madan (2006). These approaches do not allow one to easily combine prespecified marginal processes.

From the perspective of combining prespecified Lévy processes into a joint Lévy process we refer to the Lévy copula approach, Cont and Tankov (2004), Kallsen and Tankov (2006). Such an approach, though potentially feasible has dependency parameters that are difficult to relate to the widely understood structure of asset correlations. Furthermore the approach has not as yet been statistically tested on a pair of assets and we seek methods that are readily applicable to a larger number of assets.

The analysis of credit derivatives has seen a variety of dependence structures introduced to marginal processes and we mention Albrecher, Ladoucette and Schoutens (2007), Moosbrucker (2006) and Baxter (2008). Often the dependence here is one dimensional in the factor analytic sense.

In this paper we rely instead, on the observation that many univariate Lévy processes may be expressed as Brownian motion with drift, time changed by a subordinator. This is true for the variance gamma, VG, (Madan and Seneta (1990), Madan, Carr and Chang (1998)), the normal inverse Gaussian, NIG, (Barndorff-Nielsen (1998)), the hyperbolic (Eberlein and Keller (1995)), the generalized hyperbolic (Eberlein (2001)), and has recently been shown to be true for the $CGMY$ (Carr, Geman, Madan and Yor (2002)) and the Meixner process (Schoutens (2002)) by Madan and Yor (2008). It is natural to try and introduce dependence by merely correlating the Brownian motions that are being time changed.

We recognize however that as the independent time changes are active at a disjoint set of times the correlation of the Brownian motions may be of little help in continuous time and the resulting processes may remain independent. For this reason we report in this paper on the performance of this approach as a way of correlating the unit time random variables resulting from the Lévy processes and do not construct a joint process in continuous time. It is at this writing, an open question as to whether the joint density we construct is in fact infinitely divisible and thereby associated with some multidimensional Lévy process.

We first evaluate the model statistically at the level of explaining pairwise joint daily returns. We show in this context that once the marginal laws have been estimated by the univariate densities, the pairwise joint law just requires an estimation of the correlation between the two Brownian motions that were marginally subjected to a time change. It is shown that this correlation between the two Brownian motions is in absolute value always greater than the sample correlation between the asset returns. The effect of the time change is to automatically reduce the observed correlation and this effect is greater, the greater the volatility of the time changes. Given the parameters of the marginal processes, one may estimate the pairwise correlation of the two Brownian motions from a simple moment equation involving as input the sample correlation between the two assets and the parameters of the marginal distributions.

For an application employing the methods developed here to the valuation of multiasset financial contracts we refer the reader to Madan (2009) where the dependence model of this paper is used to price options on a basket of stocks. Madan (2009) correlates the top 50 stocks of the S&P 500 index using the methods of this paper and proceeds to price and hedge the cash flow to call options on the basket.

We take data on two auto stocks, Ford and GM, ten stocks in the technology sector, six stocks in the financial sector and seven industrial stocks for the period 1/4/2002 to 6/18/2008 with 1615 daily return observations and estimate the marginal laws for daily returns in the variance gamma class. The gamma time change is one of the simpler time changes with full access to the density, characteristic function and Lévy measure in terms of elementary functions. We then use the moment equation, pairwise across assets, to infer the implied correlations between the Brownian motions driving the assets. We present both the raw correlation between the returns and the higher correlation implied between the Brownian motions. Finally we present chisquare tests of model performance in terms of $p - values$ for the time changed model, and the Gaussian model as a benchmark. We conclude that the model makes a significant improvement in explaining the pairwise joint structure of daily asset returns. For the technology, and industrial sectors we also evaluate the performance of the implied multivariate model on a sample of randomly generated portfolios. We find that the model is quite capable of explaining long-short portfolio returns but is not adequate to the task of long only portfolios. The latter appears to require a more complex modeling of dependence. This departure from the model is also observed to be more pronounced the larger the number of stocks in the long only portfolio.

The outline of the paper is as follows. Section 1 presents the model in its general context along with its properties and the moment equation to be used in the estimation of dependence. The particular case of the variance gamma is developed in greater detail. Section 2 describes the data and the estimation of marginal VG laws along with the results. The two sets of correlations are presented in section 3, while the performance evaluation and chisquare tests are provided in Section 4. Section 5 reports on the empirical evaluation of portfolio return distributions implied by the model. Section 6 concludes.

1 Time Changed Brownian Motion and Dependence Modeling

For the purposes of modeling single asset return distributions that have been centered we consider zero mean univariate Lévy processes $(X_i(t), t \geq 0)$, starting at $X_i(0) = 0$, that may be represented as Brownian motions with drift time changed by a subordinator, i.e. an increasing process with independent and identically distributed invrements. We denote the subordinators by $(G_i(t), t \geq$

0) and we suppose that they have unit expectation at unit time. In addition we suppose that there exist constants $\theta_i, \sigma_i > 0$, such that

$$X_i(t) = \theta_i(G_i(t) - t) + \sigma_i W_i\left(G_i(t)\right)$$

for Brownian motions $(W_i(t), t \geq 0)$ that are independent of the subordinators $G_i(t)$.

A particular special case of this structure is the Variance Gamma model (Madan and Seneta (1990), Madan, Carr and Chang (1998)) where the processes $G_i(t)$ are gamma processes with unit mean rate and variance rate ν_i. The density of the level of the process at unit time is then the gamma density given by

$$f(x) = \frac{1}{\nu_i^{\frac{1}{\nu_i}} \Gamma\left(\frac{1}{\nu_i}\right)} x^{\frac{1}{\nu_i}-1} e^{-\frac{x}{\nu_i}}.$$

There are many other examples of such Lévy processes used for modeling financial asset returns that were cited in the introduction. We now consider introducing dependence between thees random variables at unit time by merely correlating the Brownian motions and keeping the time changing subordinators independent. We may then write at unit time that $X_i = X_i(1)$ is

$$X_i \overset{(d)}{=} \theta_i\left(G_i - 1\right) + \sigma_i \sqrt{G_i} Z_i$$

where $G_i = G_i(1)$ and the variables Z_i are standard normal variates with correlations ρ_{ij} between Z_i and Z_j for $i \neq j$. We shall refer to this model of dependence as the multiply time changed multivariate normal model.

There is now dependence between the unit returns as the covariances

$$E\left[X_i X_j\right] = \sigma_i \sigma_j E\left[\sqrt{G_i}\right] E\left[\sqrt{G_j}\right] \rho_{ij} \tag{1}$$

are not zero.

The sample correlations between the asset returns are given by

$$\widetilde{\rho}_{ij} = \frac{E\left[X_i X_j\right]}{\sqrt{E\left[X_i^2\right] E\left[X_j^2\right]}}$$

Theorem 1 *The sample correlations $\widetilde{\rho}_{ij}$ are dominated in absolute value by the correlations between the Brownian motion ρ_{ij}.*

Proof. Consider first the case of positive sample covariance. In this case

$$\begin{aligned} \widetilde{\rho}_{ij} &= \frac{E\left[X_i X_j\right]}{\sqrt{E\left[X_i^2\right] E\left[X_j^2\right]}} \\ &< \frac{E\left[X_i X_j\right]}{\sigma_i \sigma_j} \\ &< \frac{E\left[X_i X_j\right]}{\sigma_i \sigma_j E\left[\sqrt{G_i}\right] E\left[\sqrt{G_j}\right]} \\ &= \rho_{ij} \end{aligned}$$

The last inequality follows from the concavity of the square root function and the fact that

$$E\left[\sqrt{G_i}\right] \leq \sqrt{E\left[G_i\right]} = 1$$

A similar argument holds for negative covariances. ■

We learn from Theorem 1 that sample correlations understate the real correlations between the Brownian motions and this understatement is greater, the larger the time change volatilities. We also see from equation (1) that once we have estimated the marginal laws and have the specification of the time change and the parameters θ_i, σ_i, and the parameters for the subordinators we may estimate the correlation between the Brownian motions implied by the time change model by

$$\rho_{ij} = \frac{E\left[X_i X_j\right]}{\sigma_i \sigma_j E\left[\sqrt{G_i}\right] E\left[\sqrt{G_j}\right]}$$

as the numerator is estimated by evaluating a sample covariance and we need to just compute the expectation of the square root of our subordinators.

In the particular case of the variance gamma model the density of the time change at unit time has a single parameter ν_i and

$$\begin{aligned} E\left[\sqrt{G_i}\right] &= \int_0^\infty \frac{1}{\nu_i^{\frac{1}{\nu_i}} \Gamma(\frac{1}{\nu_i})} \sqrt{x} x^{\frac{1}{\nu_i}-1} e^{-\frac{x}{\nu_i}} dx \\ &= \frac{\sqrt{\nu_i}\Gamma\left(\frac{1}{\nu_i}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{\nu_i}\right)}. \end{aligned}$$

Hence once we have estimated the marginal distribution parameters $\sigma_i, \nu_i, \theta_i$ by maximum likelihood on time series data on daily returns, we estimate the correlation implied between the Brownian motions. The latter correlation is what is needed to simulate observations from the joint law proposed by the model.

2 Data and Marginal Laws

We took time series data on stock prices from 4/1/2002 to 6/18/2008 for the following tickers, F, GM in the automobile sector. For the technology sector we considered 10 names and they were $AAPL$, $AMZN$, $CSCO$, $DELL$, IBM, $INTC$, $MSFT$, $ORCL$, $QCOM$, and $YHOO$. For the financial sector we took BAC, C, GS, MER, LEH, and MS. Finally we considered the following stocks as a group, XOM, SUN, XRX, WMT, VZ, MMM, and KO. In total we had 25 stocks. For each stock we standardized the daily returns to a zero mean and unit variance and then binned the data into 100 evenly spaced bins in the the interval ± 5 standard deviations and maximized the log likelihood of the binned data following the methods of Carr, Geman, Madan and Yor (2002). The estimated VG parameters are presented for all 25 stocks in Table

1. The parameters are for daily returns and need to be scaled for their annual counterparts, but as we shall use them at the daily level, we present them without scaling. The value of θ is given in basis points as these values are small representing daily third moments. For typical graphs displaying the quality of the fit by these estimation methods we refer the reader to Madan (2006). For a sample of goodness of fit tests associated with these densities we refer to Carr, Geman, Madan and Yor (2002).

TABLE 1

VG parameter estimates
for the period 1/4/2002 to 6/18/2008
using daily log price relative returns

TICKER	σ	ν	θ in basis points
AAPL	0.0257	0.5737	13.6183
AMZN	0.0287	1.1043	30.6629
BAC	0.0166	2.7696	-22.1156
C	0.0201	2.4699	0.0004
CSCO	0.0218	0.7300	-9.5081
DELL	0.0188	0.7543	0.2387
F	0.0237	0.6088	25.1879
GM	0.0238	0.9076	24.0957
GS	0.0179	0.5790	0.0352
IBM	0.0146	0.8653	0.0167
INTC	0.0224	0.6473	-1.5322
KO	0.0109	0.7669	-0.2736
LEH	0.0275	2.6239	-31.2588
MER	0.0200	0.8421	0.0410
MMM	0.0126	0.8760	-0.0734
MS	0.0213	0.9177	-0.0457
MSFT	0.0202	2.7847	-23.4115
ORCL	0.0235	1.0347	-0.0021
QCOM	0.0239	0.6561	29.4362
SUN	0.0200	0.3400	-52.9221
VZ	0.0157	0.7489	-0.7597
WMT	0.0133	0.5167	12.1704
XOM	0.0140	0.4373	-37.9893
XRX	0.0220	0.9247	0.0506
YHOO	0.0349	3.1129	-0.0023

3 Sample and Implied Correlations

For each of the four sectors, auto, technology, financial and industrial we computed the pairwise sample correlations and the implied correlations between the Brownian motions. We present four matrices for each of the four sectors, with the sample correlation matrix in the upper diagonal and the higher implied correlation between the Brownian motions in the lower diagonal. The matrices are respectively, for the four sectors of dimension 2, 10, 6 and 7.

	F	GM
F	1	0.6270
GM	0.7901	1

	$AAPL$	$AMZN$	$CSCO$	$DELL$	IBM	$INTC$	$MSFT$	$ORCL$	$QCOM$	$YHOO$
$AAPL$	1	.2535	.3293	.3472	.3245	.3529	.2195	.2848	.2694	.2180
$AMZN$	.4009	1	.3522	.3517	.3089	.3294	.1966	.2809	.2675	.3587
$CSCO$	.4885	.4956	1	.5514	.5347	.6228	.3712	.5197	.4351	.3854
$DELL$	.5065	.4864	.7156	1	.5072	.5768	.3429	.4587	.4023	.3451
IBM	.4894	.4418	.7173	.6691	1	.5674	.3657	.5034	.3649	.3228
$INTC$	.5196	.4599	.8158	.7428	.7554	1	.3887	.5267	.4416	.3888
$MSFT$	.4437	.3768	.6676	.6064	.6684	.6937	1	.3469	.2702	.2242
$ORCL$	.4133	.3865	.6709	.5820	.6604	.6745	.6099	1	.3489	.3130
$QCOM$	.4662	.4391	.6698	.6089	.5709	.6747	.5667	.5252	1	.2859
$YHOO$	.3911	.6102	.6151	.5416	.5238	.6158	.4877	.4885	.5323	1

	BAC	C	GS	MER	LEH	MS
BAC	1	.4711	.4079	.3387	.4400	.4381
C	1	1	.7014	.5332	.7209	.7148
GS	.8012	.9341	1	.6521	.7975	.7955
MER	.9203	.9824	1	1	.6356	.6137
LEH	.8991	.9993	.9922	1	1	.7967
MS	.9055	1	1	1	1	1

	XOM	SUN	XRX	WMT	VZ	MMM	KO
XOM	1	.4354	.2654	.3692	.4165	.2816	.3727
SUN	.6467	1	.1329	.1182	.1640	.1159	.1007
XRX	.3449	.2165	1	.2678	.3002	.1791	.2143
WMT	.4339	.1741	.3449	1	.3955	.2878	.3327
VZ	.5138	.2529	.4051	.4825	1	.2081	.3305
MMM	.5891	.3036	.4106	.5967	.4517	1	.2352
KO	.4689	.1586	.2955	.4148	.4314	.5218	1

We see from these matrices the degree of understatement in sample correlations induced by the non-Gaussian nature of the marginal distributions. For the financial sector some implied correlations were slightly above unity and these were truncated at unity in the matrices being presented. We checked that excepting the implied correlation matrix for the financial sector where we had to truncate values above unity, all the remaining matrices had all positive eigenvalues and were positive definite matrices. The financial sector has been particularly disturbed over this time period and given the absence of strictly positive definite matrices here we exclude it from the performance study reported in the next section.

4 Performance Evaluation

We wish to evaluate whether the model describing unit time returns as multivariate Brownian motion taken at independent gamma subordinators evaluated at unit time makes an improvement against the benchmark of a purely Gaussian model. For this purpose we took our stocks pairwise in the three groups of auto, technology and industrial and we computed by simulation over 100000 readings the expected number of observations in a number of cells for both the benchmark Gaussian model and the alternate model proposed here. We then counted the observed number in each of the cells with an expected number exceeding 2% and 1% of the total number of observations in the time series. From these we computed two chisquare statistics, one for the benchmark and one for the alternate model (Mood and Graybill (1974)). We also computed $p-values$ for both models. A high chisquare statistic with an associated low $p-value$ reflects a poor explanation of the empirically observed joint distribution of returns by the associated model.

We report the chisquare statistics in three matrices, for the three sectors, with the upper diagonal for the Gaussian model and lower diagonal for the correlated time change model. Since the $p-values$ for the Gaussian model and also the time changed model under independence are zero we do not report these. Instead we report $p-values$ only for the correlated time change model with the upper diagonal containing the less stringent test for cells with expected outcomes in excess of 2% of the observations while the lower diagonal is the more stringent test with expected outcomes in excess of 1% of the observations. There are then three matrices for the $p-values$. The matrix dimensions are now 2, 10, and 7 for the three sectors reported. We first present the chisquare statistics followed by the $p-values$.

chisquares	*F*	*GM*
F	0	390.6481
GM	17.2854	0

chisquares	*AAPL*	*AMZN*	*CSCO*	*DELL*	*IBM*	*INTC*	*MSFT*	*ORCL*	*QCOM*	*YHOO*
AAPL	0	384.61	329.10	302.14	398.79	329.93	832.99	288.80	445.78	597.57
AMZN	18.72	0	324.42	312.77	403.43	337.80	822.63	365.25	496.64	784.37
CSCO	5.67	20.69	0	300.28	448.09	513.44	951.88	354.17	438.46	598.90
DELL	24.26	21.99	20.17	0	329.65	387.29	777.54	281.06	370.39	462.63
IBM	20.27	24.29	24.62	11.56	0	461.33	1056.67	411.56	465.87	564.98
INTC	19.00	30.52	32.87	19.38	19.94	0	907.52	427.22	425.41	527.02
MSFT	129.64	164.99	197.02	184.16	183.23	229.99	0	966.75	1038.19	1226.04
ORCL	13.91	18.62	13.15	25.58	25.91	31.09	202.72	0	432.75	494.15
QCOM	17.37	38.08	25.23	17.67	26.52	26.69	152.09	32.22	0	753.19
YHOO	164.41	161.21	125.50	127.06	122.97	158.52	241.18	111.40	164.81	0

$chisquares$	XOM	SUN	XRX	WMT	VZ	MMM	KO
XOM	0	312.94	170.17	108.08	155.43	725.49	159.11
SUN	24.47	0	252.59	169.31	214.96	906.56	212.06
XRX	17.77	10.24	0	213.64	321.58	983.21	286.81
WMT	12.13	20.04	21.80	0	217.10	840.04	204.90
VZ	16.69	18.98	39.10	39.87	0	959.72	332.13
MMM	21.23	15.24	29.19	11.56	31.28	0	920.82
KO	24.48	25.08	42.08	31.95	50.99	33.30	0

$p-vals$	F	GM
F	0	.1392
GM	.0033	0

$p-vals$	$AAPL$	$AMZN$	$CSCO$	$DELL$	IBM	$INTC$	$MSFT$	$ORCL$	$QCOM$	$YHOO$
$AAPL$	0	.1757	.9912	.1124	.2083	.3284	0	.5327	.2970	0
$AMZN$	.2883	0	.0550	.0377	.0186	.0039	0	.0982	.0009	0
$CSCO$	.8036	.0017	0	.1247	.0385	.0018	0	.5145	.0469	0
$DELL$	.3379	.1327	.0023	0	.7122	.1509	0	.0825	.3435	0
IBM	.2267	.000036	.00012	.4289	0	.0966	0	.0265	.0222	0
$INTC$	.1429	.00035	.0000052	.0238	.000041	0	0	.0085	.0313	0
$MSFT$	0	0	0	0	0	0	0	0	0	0
$ORCL$	.1068	.00012	.00022	.0011	.000323	.0000013	0	0	.0037	0
$QCOM$	.0711	.0015	.000133	.0279	.001711	.0033	0	.000122	0	0
$YHOO$	0	0	0	0	0	0	0	0	0	0

$p-vals$	XOM	SUN	XRX	WMT	VZ	MMM	KO
XOM	0	.0575	.2749	.7347	.3375	.1696	.0574
SUN	.0843	0	.7442	.2182	.1658	.3618	.0338
XRX	.4729	.3557	0	.1132	.0017	.0328	.00038
WMT	.2549	.0394	.3663	0	.0008	.7122	.0102
VZ	.1062	.5394	.000038	.0053	0	.0184	.000029
MMM	.1324	.1657	.00044	.5767	.0269	0	.0103
KO	.0008	.0309	.000099	.0029	.0000022	.0124	0

We make the following remarks on these results.

- The multiply time changed multivariate normal model is clearly a significant improvement over a simple Gaussian correlated model.
- Though in some cases the larger number of cells are better explained by the multiply time changed multivariate normal model, this is generally not the case.

- The multiply time changed multivariate normal model could be further improved, especially if one wants a better description of tails.
- Minimally, investigations of dependence could employ the multiply time changed multivariate normal model.
- Interestingly, modeling the dependence between all the assets with $MSFT$ and $YHOO$ proved particularly difficult. However, even here we observe from the chisquare statistics the magnitude of the improvement made by the multiply time changed multivariate normal model.

5 Performance on Portfolio returns

We now evaluate the ability of the multiply time changed multivariate normal model to explain portfolio returns. We restrict attention to the technology sector, excluding for this purpose both $MSFT$ and $YHOO$ and the industrial sector. So we have eight stocks from technology and seven stocks from our so-called industrial sector. We may generate by simulation a large number (we use 100000) of returns consistent with the model. For each of a randomly selected set of portfolios we may construct returns consistent with the model and build a histogram of expected outcomes in a variety of cells. From the time series data we determine the observed number of portfolio returns in the same cells and then evaluate a chisquare statistic and a $p - value$ for each portfolio. We report the results of such an experiment in this section.

We formed for each of the two sectors technology and industrial a thousand randomly selected long-short portfolios and a thousand long only portfolios. For the long-short portfolio of n assets we generated n independent normal variates and scaled them to be on the unit n sphere. For the long only we generated n independent normals and scaled the absolute values by their sum. For each portfolio we constructed the simulated return from the model and determined the expected number in 21 cells in the 5% to 95% quantiles. From the time series we determined the portfolio returns in the data. We then constructed the chisquare statistic and $p - values$. Finally we constructed empirical complementary distribution function of the $p-values$ across the 1000 portfolios for each sector and each type of portfolio. We present in Figure (1) a graph of the complementary distribution functions that describes the proportion of portfolios with a $p - value$ greater than the target $p - value$ given on the $x - axis$. A dominating complementary distribution function thereby reflects a superior model in its ability to explain the univariate laws of arbitrary selected portfolios.

We see from these graphs that the $p-values$ are consistently higher for long-short portfolios and this suggests that the dependence embedded in long only portfolios is much harder to capture for the multiply time changed multivariate normal model. The performance with respect to long-short portfolios is quite adequate.

With a view to possibly detecting a size effect in the departure of long-only portfolios from the model we combined the two sectors of 8 technology

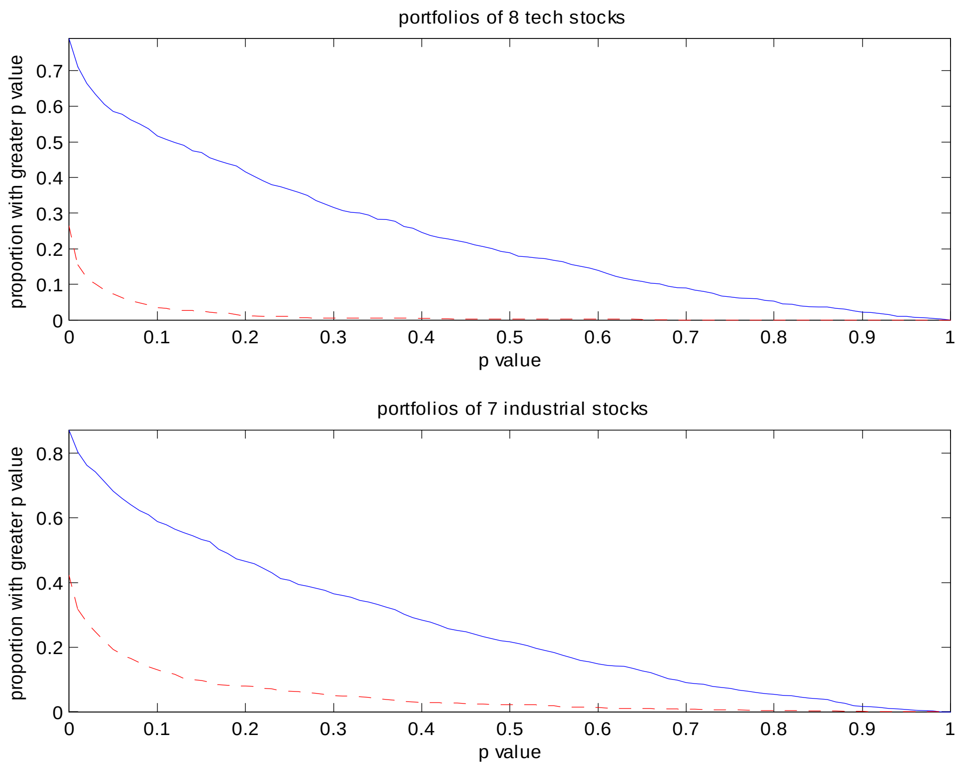

Figure 1: Long-Short portfolio complementary distribution function of p-values in blue. Long only portfolios are in red.

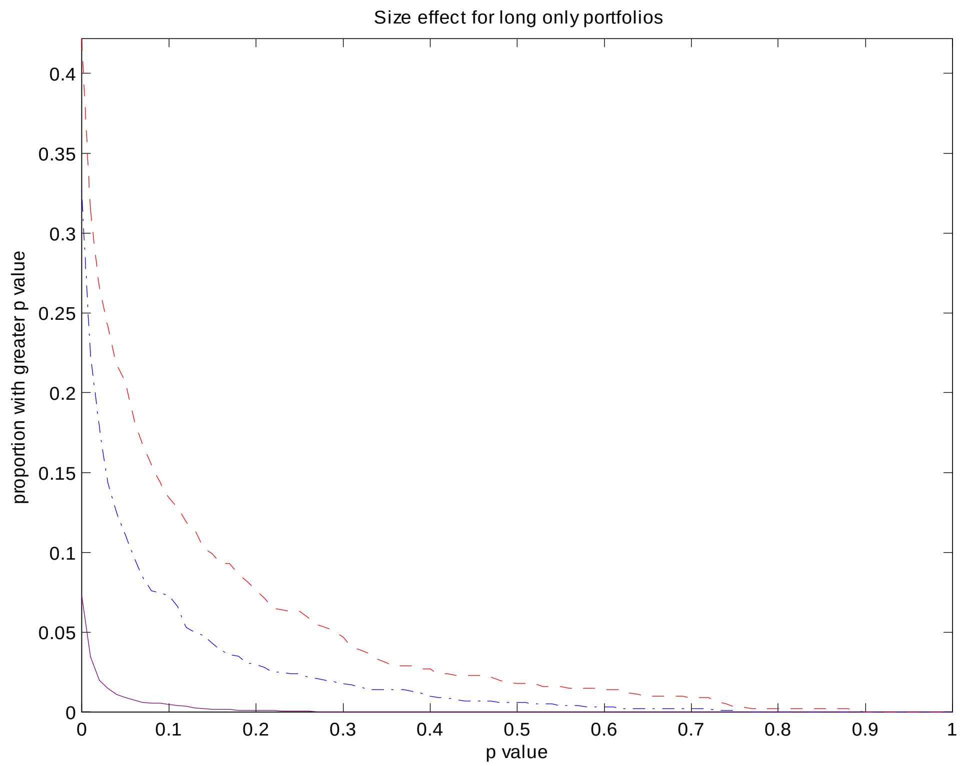

Figure 2: The 8 technology stocks complementary distribution function (CDF) of p values for long only portfolios is shown in blue. The 7 industrial stock portfolios p value CDF is shown in red. The 15 stock portfolio p-value CDF is in black

stocks and 7 industrial stocks and formed long-only portfolios with 15 stocks and recomputed the $p-values$ for such portfolios. We present in Figure (2) the complementary distribution function of the 15 stock $p-values$ compared with the two sets of 8 stock and 7 stock portfolios. There appears to be a clear size effect associated with the departure of long-only portfolios from the model.

6 Conclusion

We develop a relatively simple approach to correlating the unit period returns resulting from Lévy processes. The basic idea is to write the Lévy process as a time changed Brownian motion and to correlate the Brownian motions. We show in this context that sample correlations understate the correlation between the Brownian motions and develop an expression that uses the laws of the marginal time changes to correct the sample correlation upwards.

We perform tests of pairwise modeling of dependence between stocks and show that the model makes a significant and quite adequate improvement over Gaussian correlations in describing the dependence. We therefore advocate that at a minimum one should use the correlated time change model over Gaussian models of dependence. The model could be further improved particular with respect to enhancing the modeling of dependence in the tails.

We also evaluate the model from the perspective of explaining portfolio returns in larger dimensions and find that we have some degree of adequacy for long-short portfolios. The long only portfolios appear to require a more complex modeling of dependency. We leave these questions for future research.

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