The Lévy swap market model

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Abstract

Models driven by Lévy processes are attractive since they allow for better statistical fitting compared to classical diffusion models. We derive the dynamics of the forward swap rate process in a semimartingale setting and introduce a Lévy swap market model. In order to guarantee positive rates, we model the swap rates as ordinary exponentials. We start with the most distant rate which is driven by a non-homogeneous Lévy process. Via backward induction we construct the remaining swap rates such that they become martingales under the corresponding forward swap measures. Finally we show how swaptions can be priced using bilateral Laplace transforms.

1 Introduction

The market models of interest rate dynamics have become increasingly popular among practitioners. These models directly specify the arbitrage-free dynamics of a set of forward Libor or swap rates. The models for forward swap rates in a pure diffusion (Brownian motion) setting were developed by Jamshidian (1997), Musiela and Rutkowski (1997b) and Rutkowski (1999, 2001). More recent approaches to interest rate models involve jump-diffusions and more generally, models driven by Lévy processes. The latter are becoming increasingly popular in finance since they allow for better statistical fitting compared to classical diffusion models (see Eberlein (2001), Eberlein and Kluge (2006), Eberlein and Koval (2005)). Processes that include jumps have already been used in several papers describing the modelling of various interest rates. In the context of instantaneous, continuously compounded interest rates, Björk, Kabanov and Runggaldier (1997) extend the classical Heath, Jarrow and Morton (1992) (henceforth HJM) framework to the case of diffusion-multivariate point processes, and Björk, Di Masi, Kabanov and Runggaldier (1997) to general semimartingales. Glasserman and Kou (2003) characterized the arbitrage-free dynamics of interest rates when the term structure is modelled through forward Libor rates or forward swap rates, in the presence of both jumps and a diffusion process. They consider the case when the jumps are modelled through a finite number of marked point processes, in which case the purely discontinuous part is of bounded variation. More explicitly, they place themselves in the generalized HJM framework of Björk, Kabanov and Runggaldier (1997) and show that the simple forward rates can be embedded in an arbitrage-free model of instantaneous forward rates. Eberlein and Özkan (2005) push the forward

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Libor rate approach further into a setting of more general jump processes. Apart from the HJM framework of Björk, Di Masi, Kabanov and Runggaldier (1997), they consider the Lévy setting of Eberlein and Raible (1999) and construct the discrete tenor Lévy Libor model directly through backward induction, whence extending the approach of Musiela and Rutkowski (1997a, 1997b) from the case of pure diffusions to the Lévy setting. Eberlein and Kluge (2006) price swaptions in the HJM framework using non-homogeneous Lévy processes.

In this paper, we will develop a market model of the forward swap rates by allowing the driving process to be a Lévy process. In that sense we generalize the corresponding result in Glasserman and Kou (2003). However, our approach differs in a crucial way from that of Glasserman and Kou (2003) in that we do not start by showing that the forward swap rate model can be embedded in the framework of instantaneous forward rates of Björk, Di Masi, Kabanov and Runggaldier (1997) or Eberlein and Raible (1999). In fact, as pointed out in Hunt and Kennedy (2000), the extra burden of proving that the models fall within the HJM class is unnecessary. Instead, we use a numéraire-based approach and do not explicitly specify the dynamics of the instantaneous forward rates or the bond prices. The outline of such a modelling approach in a pure diffusion setting can be found in Pelsser (2000), and in Hunt and Kennedy (2000). Furthermore, we extend the backward induction method of Rutkowski (1999, 2001) to the case where the forward swap rates are driven by a non-homogeneous Lévy process.

This paper is organized as follows. In Section 2 we consider the so-called regular and reverse swap market models. We extend the approach given in Hunt and Kennedy (2000), and in Pelsser (2000), from the pure diffusion setting to a more general semimartingale setting and derive the condition for those models to be arbitrage-free. In Section 3 we develop the model for forward swap rates. This setting is appropriate for practical implementations. We discuss the pricing of plain vanilla swaptions in Section 4 by using Laplace transforms.

Finally we mention that background material about Lévy processes can be found in the textbooks of Applebaum (2004), Bertoin (1996) and Sato (1999), and in a general semimartingale setting in Jacod and Shiryaev (1987). A good reference for applications of Lévy processes is Schoutens (2003).

2 Swap market models under a single measure

In this section we set up a market model for forward swap rates by specifying the arbitrage-free dynamics of the rates under a single measure. This is a useful approach if we want to determine the price of more sophisticated derivatives. Assume we are given a collection of dates (called tenor structure) $0 < T_0 < T_1 < \ldots < T_N$, and associated to each tenor date T_i is a zero-coupon bond that matures at this date. We denote the price at time t of such a bond by $B(t, T_i)$.

We extend the approach given in Hunt and Kennedy (2000), and in Pelsser (2000), in order to find the arbitrage-free dynamics for the forward swap rates under the T_N-forward measure $\mathbb{P}_N$ (also called terminal measure [2]), and under the T_0-forward measure $\mathbb{P}_0$. As mentioned in Section 1 this approach is numéraire-based. We start by specifying the dynamics for the forward swap rates and then determine the necessary relationship for any corresponding term structure model to be arbitrage-free.

Note that in this section we do not explicitly assume that the driving process is a Lévy process. However, the latter can be embedded in the given setting.

We assume in the sequel that we are given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$, such that the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ satisfies the

[2] This is the measure associated to the numéraire $B(t, T_N)$.

usual conditions, cf. Jacod and Shiryaev (1987).

2.1 Regular swap market model

We assume that the tenor structure $0 < T_0 < T_1 < \ldots < T_N$ is given, and $\delta_i = T_i - T_{i-1}$ for $i = 1, \ldots, N$, where T_N is a fixed time horizon. δ_i is the fraction of a year corresponding to the period from T_{i-1} to T_i. It represents the length of the ith accrual period (typically a month, a quarter, or a year). We consider a family of forward swap rates $S_i(t) := S(t, T_i, T_N)$, $t \in [0, T_i)$, which have the same maturity date T_N for all $i = 0, \ldots, N-1$. Given a family of zero-coupon bond prices $(B(t, T_j))_{0 \le t \le T_j}$ for $j = 0, \ldots, N$ which are assumed to be positive semimartingales, but whose dynamics does not have to be specified any further, the accrual factor for any individual swap rate from such a family is given by

$$C_{i,N}(t) := \sum_{j=i+1}^{N} \delta_j B(t, T_j), \tag{1}$$

and the forward swap rate is defined as

$$S_i(t) = \frac{B(t, T_i) - B(t, T_N)}{C_{i,N}(t)}, \tag{2}$$

(see e.g. Brigo and Mercurio (2001) and Pelsser (2000) for more details). Musiela and Rutkowski (1997b) define a probability measure $\widetilde{\mathbb{P}}_i = \widetilde{\mathbb{P}}_{T_i}$, which is equivalent to the historical probability measure $\mathbb{P}$, to be the forward swap measure associated with the dates T_i and T_N. Under this measure the relative bond prices $\frac{B(t,T_l)}{\delta_i B(t,T_i) + \cdots + \delta_N B(t,T_N)}$, ($t \in [0, T_l \wedge T_i]$), are local martingales under $\widetilde{\mathbb{P}}_i$ for every $l = 0, \ldots, N$.

By choosing the bond with the largest maturity T_N to be a numéraire, the discounted accrual factor (w.r.t. that numéraire) reads

$$P_t^i := \sum_{j=i+1}^{N} \frac{\delta_j B(t, T_j)}{B(t, T_N)} = \frac{C_{i,N}(t)}{B(t, T_N)}, \tag{3}$$

which should become a local martingale under the forward measure $\mathbb{P}_N$. We also define for $0 \le i \le N-1$ the following product

$$\Psi_t^i := \prod_{j=0}^{i} \left(1 + \delta_{j+1} S_{j+1}(t)\right). \tag{4}$$

We follow the convention that empty sums and products are zero and one, respectively. Note that $P_t^N \equiv S_N(t) \equiv 0$.
We can rewrite (3) as follows by using (2),

$$\begin{aligned}
\frac{C_{i,N}(t)}{B(t,T_N)} &= \frac{\delta_{i+1} B(t, T_{i+1}) + (\delta_{i+2} B(t, T_{i+2}) + \cdots + \delta_N B(t, T_N))}{B(t, T_N)} \\
&= \delta_{i+1} + \delta_{i+1} \left(\frac{B(t, T_{i+1}) - B(t, T_N)}{C_{i+1,N}(t)} \cdot \frac{C_{i+1,N}(t)}{B(t, T_N)} \right) + \frac{C_{i+1,N}(t)}{B(t, T_N)} \\
&= \delta_{i+1} + (1 + \delta_{i+1} S_{i+1}(t)) \frac{C_{i+1,N}(t)}{B(t, T_N)}.
\end{aligned}$$

By substituting the P_t^i from (3) into the above equation, we obtain for $i = 0, \dots, N-1$ the recursive relation

$$P_t^i = \delta_{i+1} + (1 + \delta_{i+1} S_{i+1}(t)) \cdot P_t^{i+1}. \tag{5}$$

Multiplying both sides of the equation (5) by Ψ_t^{i-1} we obtain

$$\Psi_t^{i-1} P_t^i = \delta_{i+1} \Psi_t^{i-1} + \Psi_t^i P_t^{i+1}.$$

By backward induction, down from $i = N-1$, it follows that

$$P_t^i = \frac{\sum_{j=i}^{N-1} \delta_{j+1} \Psi_t^{j-1}}{\Psi_t^{i-1}} = \sum_{j=i}^{N-1} \delta_{j+1} \prod_{k=i+1}^{j} (1 + \delta_k S_k(t)). \tag{6}$$

Remark 2.1 *The P_t^i in* (6) *corresponds to the notation $s_{ii}(t)$ in Jamshidian (1997). More generally, Jamshidian (1997) defines a sum*

$$s_{ij}(t) = \sum_{k=j}^{N-1} \delta_{k+1} \prod_{l=i+1}^{k} (1 + \delta_l S_l(t)),$$

which is equivalent to $\frac{\Psi_t^{j-1}}{\Psi_t^{i-1}} P_t^j$.

Our aim is to develop an arbitrage-free model for the term structure of interest rates specified through forward swap rates $S_i(\cdot)$. The next theorem states the swap rate model under the terminal measure $\mathbb{P}_N$. This theorem slightly generalizes Theorem 5.1 in Glasserman and Kou (2003) and is an extension of the approach in Hunt and Kennedy (2000, Section 18.3) from the Gaussian driving process to a more general setting of jump processes.

Theorem 2.1 *For each $i = 0, \dots, N-1$, let $\theta_i(\cdot)$ be a bounded $\mathbb{R}^d$-valued function and $G_i : \mathbb{R}_+ \times \mathbb{R}^r \to (-1, \infty)$ be a deterministic function in $G_{\text{loc}}(\mu)$* [3]. *Let W^N be a standard Brownian motion in $\mathbb{R}^d$ with respect to $\mathbb{P}_N$, and μ the random measure of jumps of a semimartingale with the continuous compensator $\nu^N(dt, dx) = \lambda^N(t, dx)dt$. The dynamics of $S_i(\cdot)$, $i = 0, \dots, N-1$, is assumed to satisfy*

$$\frac{dS_i(t)}{S_i(t-)} = \alpha_i(t)dt + \theta_i(t)dW_t^N + \int_{\mathbb{R}^r} G_i(t, x)(\mu - \nu^N)(dt, dx). \tag{7}$$

Then this model is arbitrage-free if

$$\begin{aligned} \alpha_i(t) = &-\sum_{j=i+1}^{N-1} \frac{\delta_j \sum_{k=j}^{N-1} \delta_{k+1} \prod_{l=i+1}^{k} (1 + \delta_l S_l(t-)) S_j(t-) \theta_j(t) \theta_i(t)}{\sum_{k=i}^{N-1} \delta_{k+1} \prod_{l=i+1}^{k} (1 + \delta_l S_l(t-)) \cdot (1 + \delta_j S_j(t-))} \\ &+ \int_{\mathbb{R}^r} G_i(t, x) \left[1 - \frac{\sum_{j=i}^{N-1} \delta_{j+1} \prod_{k=i+1}^{j} \left(1 + \delta_k S_k(t-)(1 + G_k(t, x))\right)}{\sum_{j=i}^{N-1} \delta_{j+1} \prod_{k=i+1}^{j} (1 + \delta_k S_k(t-))} \right] \lambda^N(t, dx). \end{aligned} \tag{8}$$

Proof. As we want the model (7) to be arbitrage-free, each of the P_t^i, $i = 0, \dots, N-1$, defined in (3), has to become a local martingale under the measure $\mathbb{P}_N$. This imposes a relationship between the finite variation term and the coefficients of the Gaussian and the discontinuous terms in (7),

[3] For definition of this set we refer to Jacod and Shiryaev (1987, II.1.27)

which we now derive.
Applying Itô's formula (in its product rule's version) to (5) leads to

$$dP_t^i = (1+\delta_{i+1}S_{i+1}(t-))\,dP_t^{i+1} + \delta_{i+1}P_{t-}^{i+1}dS_{i+1}(t) + \delta_{i+1}d\left[S_{i+1},P^{i+1}\right]_t. \tag{9}$$

The quadratic covariation term on the right-hand side of (9) is according to Jacod and Shiryaev (1987, Definition I.4.53)

$$\left[S_{i+1},P^{i+1}\right]_t = \langle S_{i+1}^c, P^{i+1,c}\rangle_t + \sum_{0\le s\le t}\Delta S_{i+1}(s)\Delta P_s^{i+1}, \tag{10}$$

where $\Delta S_{i+1}(t)$ and ΔP_t^{i+1} denote the jumps of $S_{i+1}(t)$ and P_t^{i+1}, respectively. Recall that the superscript c indicates that we consider the the continuous martingale part.
These jumps at time t follow from (7) and (6) by definition:

$$\Delta S_{i+1}(t) = \int_{\mathbb{R}^r} S_{i+1}(t-)G_{i+1}(t,x)\mu(\{t\},dx),$$

and

$$\begin{aligned}\Delta P_t^{i+1} = \int_{\mathbb{R}^r}\Bigg(&\sum_{j=i+1}^{N-1}\delta_{j+1}\prod_{k=i+2}^{j}\big(1+\delta_kS_k(t-)(1+G_k(t,x))\big)\\ &-\sum_{j=i+1}^{N-1}\delta_{j+1}\prod_{k=i+2}^{j}(1+\delta_kS_k(t-))\Bigg)\mu(\{t\},dx).\end{aligned}$$

We can now express (9) more explicitly as

$$\begin{aligned}dP_t^i =& (1+\delta_{i+1}S_{i+1}(t-))\,dP_t^{i+1} + \delta_{i+1}P_{t-}^{i+1}dS_{i+1}(t) + \delta_{i+1}d\langle S_{i+1}^c, P^{i+1,c}\rangle_t\\ &+\delta_{i+1}S_{i+1}(t-)\int_{\mathbb{R}^r}G_{i+1}(t,x)\times\\ &\times\left[\sum_{j=i+1}^{N-1}\delta_{j+1}\prod_{k=i+2}^{j}\big(1+\delta_kS_k(t-)(1+G_k(t,x))\big)-P_{t-}^{i+1}\right]\mu(dt,dx)\\ =& (1+\delta_{i+1}S_{i+1}(t-))\,dP_t^{i+1} + \delta_{i+1}P_{t-}^{i+1}dS_{i+1}(t) + \delta_{i+1}d\langle S_{i+1}^c, P^{i+1,c}\rangle_t\\ &+\delta_{i+1}S_{i+1}(t-)\int_{\mathbb{R}^r}G_{i+1}(t,x)\times\\ &\times\left[\sum_{j=i+1}^{N-1}\delta_{j+1}\prod_{k=i+2}^{j}\big(1+\delta_kS_k(t-)(1+G_k(t,x))\big)-P_{t-}^{i+1}\right](\mu-\nu^N)(dt,dx)\\ &+\delta_{i+1}S_{i+1}(t-)\int_{\mathbb{R}^r}G_{i+1}(t,x)\times\\ &\times\left[\sum_{j=i+1}^{N-1}\delta_{j+1}\prod_{k=i+2}^{j}\big(1+\delta_kS_k(t-)(1+G_k(t,x))\big)-P_{t-}^{i+1}\right]\nu^N(dt,dx).\end{aligned} \tag{11}$$

Recall that P_t^i has to become a local martingale under the measure $\mathbb{P}_N$. Eliminating the finite variation terms – which will be set to equal zero – in the SDE (11) yields when taking (7) into account

$$
\begin{aligned}
dP^i_t &= (1+\delta_{i+1}S_{i+1}(t-))\,dP^{i+1}_t + P^{i+1}_{t-}\delta_{i+1}S_{i+1}(t-)\theta_{i+1}(t)dW^N_t \\
&\quad + \delta_{i+1}P^{i+1}_{t-}S_{i+1}(t-)\int_{\mathbb{R}^r} G_{i+1}(t,x)(\mu-\nu^N)(dt,dx) \\
&\quad + \delta_{i+1}S_{i+1}(t-)\int_{\mathbb{R}^r} G_{i+1}(t,x)\times \\
&\quad \times \left[\sum_{j=i+1}^{N-1}\delta_{j+1}\prod_{k=i+2}^{j}\big(1+\delta_k S_k(t-)(1+G_k(t,x))\big) - P^{i+1}_{t-}\right](\mu-\nu^N)(dt,dx) \qquad (12)
\end{aligned}
$$

Multiplying both sides of the equation (12) by Ψ^{i-1}_{t-} we get

$$
\begin{aligned}
&\Psi^{i-1}_{t-}dP^i_t \\
&= \Psi^i_{t-}dP^{i+1}_t + \Psi^i_{t-}P^{i+1}_{t-}\left(\frac{\delta_{i+1}S_{i+1}(t-)}{1+\delta_{i+1}S_{i+1}(t-)}\right)\theta_{i+1}(t)dW^N_t \\
&+ \Psi^i_{t-}P^{i+1}_{t-}\left(\frac{\delta_{i+1}S_{i+1}(t-)}{1+\delta_{i+1}S_{i+1}(t-)}\right)\int_{\mathbb{R}^r} G_{i+1}(t,x)(\mu-\nu^N)(dt,dx) \\
&+ \Psi^i_{t-}\left(\frac{\delta_{i+1}S_{i+1}(t-)}{1+\delta_{i+1}S_{i+1}(t-)}\right)\int_{\mathbb{R}^r} G_{i+1}(t,x)\times \\
&\quad\times\left[\sum_{j=i+1}^{N-1}\delta_{j+1}\prod_{k=i+2}^{j}(1+\delta_k S_k(t-)\big(1+G_k(t,x)\big)) - P^{i+1}_{t-}\right](\mu-\nu^N)(dt,dx). \qquad (13)
\end{aligned}
$$

In order to obtain a non-recursive expression for dP^i_t we proceed by backward induction, down from $i=N-1$. For clarity, we consider a few steps of this induction procedure.

(1) In case when $i=N-1$, $\Psi^{N-2}_{t-}dP^{N-1}_t = 0$ since $P^N_t = 0$ and $S_N = 0$.

(2) If $i = N-2$ we obtain

$$
\begin{aligned}
&\Psi^{N-3}_{t-}dP^{N-2}_t \\
&= \Psi^{N-2}_{t-}dP^{N-1}_t + \Psi^{N-2}_{t-}P^{N-1}_{t-}\left(\frac{\delta_{N-1}S_{N-1}(t-)}{1+\delta_{N-1}S_{N-1}(t-)}\right)\theta_{N-1}(t)dW^N_t \\
&+ \Psi^{N-2}_{t-}P^{N-1}_{t-}\left(\frac{\delta_{N-1}S_{N-1}(t-)}{1+\delta_{N-1}S_{N-1}(t-)}\right)\int_{\mathbb{R}^r} G_{N-1}(t,x)(\mu-\nu^N)(dt,dx) \\
&+ \Psi^{N-2}_{t-}\left(\frac{\delta_{N-1}S_{N-1}(t-)}{1+\delta_{N-1}S_{N-1}(t-)}\right)\int_{\mathbb{R}^r} G_{N-1}(t,x)\times \\
&\quad\times\left[\sum_{j=N-1}^{N-1}\delta_{j+1}\prod_{k=N}^{j}\big(1+\delta_k S_k(t-)(1+G_k(t,x))\big) - P^{N-1}_{t-}\right](\mu-\nu^N)(dt,dx),
\end{aligned}
$$

where $\Psi^{N-2}_{t-}dP^{N-1}_t = 0$ from the previous step (1). This yields

$$
\Psi^{N-3}_{t-}dP^{N-2}_t = \Psi^{N-2}_{t-}P^{N-1}_{t-}\left(\frac{\delta_{N-1}S_{N-1}(t-)}{1+\delta_{N-1}S_{N-1}(t-)}\right)\theta_{N-1}(t)dW^N_t
$$

$$+ \Psi_{t-}^{N-2} \left(\frac{\delta_{N-1} S_{N-1}(t-)}{1 + \delta_{N-1} S_{N-1}(t-)} \right) \int_{\mathbb{R}^r} G_{N-1}(t,x) \times$$
$$\times \left[\sum_{j=N-1}^{N-1} \delta_{j+1} \prod_{k=N}^{j} \big(1 + \delta_k S_k(t-)(1 + G_k(t,x))\big) \right] (\mu - \nu^N)(dt, dx),$$

For the more detailed description of this induction procedure we refer to Liinev (2004).

For a general i, $i = 0, \ldots, N-1$, it therefore follows that

$$\Psi_{t-}^{i-1} dP_t^i = \sum_{j=i+1}^{N-1} \Psi_{t-}^{j-1} P_{t-}^j \left(\frac{\delta_j S_j(t-)}{1 + \delta_j S_j(t-)} \right) \theta_j(t) dW_t^N$$
$$+ \sum_{j=i+1}^{N-1} \Psi_{t-}^{j-1} \left(\frac{\delta_j S_j(t-)}{1 + \delta_j S_j(t-)} \right) \int_{\mathbb{R}^r} G_j(t,x) \times$$
$$\times \left[\sum_{l=j}^{N-1} \delta_{l+1} \prod_{k=j+1}^{l} \big(1 + \delta_k S_k(t-)(1 + G_k(t,x))\big) \right] (\mu - \nu^N)(dt, dx),$$

and thus

$$dP_t^i = P_{t-}^i \sum_{j=i+1}^{N-1} \frac{\Psi_{t-}^{j-1} P_{t-}^j}{\Psi_{t-}^{i-1} P_{t-}^i} \left(\frac{\delta_j S_j(t-)}{1 + \delta_j S_j(t-)} \right) \theta_j(t) dW_t^N$$
$$+ P_{t-}^i \sum_{j=i+1}^{N-1} \frac{\Psi_{t-}^{j-1}}{\Psi_{t-}^{i-1} P_{t-}^i} \left(\frac{\delta_j S_j(t-)}{1 + \delta_j S_j(t-)} \right) \times$$
$$\times \int_{\mathbb{R}^r} G_j(t,x) \left[\sum_{l=j}^{N-1} \delta_{l+1} \prod_{k=j+1}^{l} \big(1 + \delta_k S_k(t-)(1 + G_k(t,x))\big) \right] (\mu - \nu^N)(dt, dx), \tag{14}$$

which indeed represents a local martingale. Now we can investigate the finite variation terms in (9) and (11):

$$\delta_{i+1} P_{t-}^{i+1} \alpha_{i+1}(t) S_{i+1}(t-) dt + \delta_{i+1} d\langle S_{i+1}^c, P^{i+1,c} \rangle_t + \delta_{i+1} S_{i+1}(t-) \times$$
$$\times \int_{\mathbb{R}^r} G_{i+1}(t,x) \left[\sum_{j=i+1}^{N-1} \delta_{j+1} \prod_{k=i+2}^{j} \big(1 + \delta_k S_k(t-)(1 + G_k(t,x))\big) - P_{t-}^{i+1} \right] \nu^N(dt, dx) = 0, \tag{15}$$

where from (7) and (14)

$$d\langle S_{i+1}^c, P^{i+1,c} \rangle_t = S_{i+1}(t-) \theta_{i+1}(t) P_{t-}^{i+1} \sum_{j=i+2}^{N-1} \frac{\Psi_{t-}^{j-1} P_{t-}^j}{\Psi_{t-}^i P_{t-}^{i+1}} \left(\frac{\delta_j S_j(t-)}{1 + \delta_j S_j(t-)} \right) \theta_j(t) dt. \tag{16}$$

Combining (15) and (16) we can easily find that the drift term in (7) is for $i = 0, \ldots, N-1$ given by

$$\alpha_i(t) = -\sum_{j=i+1}^{N-1} \frac{\Psi_{t-}^{j-1} P_{t-}^{j}}{\Psi_{t-}^{i-1} P_{t-}^{i}} \left(\frac{\delta_j S_j(t-)}{1+\delta_j S_j(t-)} \right) \theta_j(t)\theta_i(t)$$
$$- \int_{\mathbb{R}^r} G_i(t,x) \left[\frac{\sum_{j=i}^{N-1} \delta_{j+1} \prod_{k=i+1}^{j} \left(1+\delta_k S_k(t-)(1+G_k(t,x))\right)}{P_{t-}^{i}} - 1 \right] \lambda^N(t,dx). \quad (17)$$

By taking into account the definition (4) and relation (6), we can express the right-hand side of (17) through forward swap rates and their volatilities, yielding (8).

□

2.2 Reverse swap market models

In this section we consider another type of forward swap rate model for so-called reverse swap markets. In the pure diffusion setting this model is developed in Hunt and Kennedy (2000), and Pelsser (2000). We show here that it is indeed possible to extend this model to a semimartingale setting. Hunt and Kennedy (2000) argue that the model in Section 2.1 is suitable for path-dependent products such as barrier swaptions where the pricing is done with respect to the collection of swap rates which are reset on different dates but have a common maturity date. In this section we consider the reverse situation where the family of swap rates to be modelled, has a common start date and different maturities. These are the swap rates which underlie for example spread options. Deriving the arbitrage-free dynamics for the forward swap rates in this case is analogous to the proof of Theorem 2.1 in Section 2.1. Therefore we do not go into details, but rather present the main steps for developing the relevant dynamics.

We consider again the tenor structure $0 < T_0 < T_1 < \cdots < T_N$, and a series of swaps starting on date $T := T_0$ having maturity dates $T_1, \ldots, T_N$, which now differ from those of Section 2.1. We define for $i = 1, \ldots, N$

$$C_{i,N}(t) := \sum_{j=1}^{i} \delta_j B(t, T_j),$$

$$S_i(t) := S_i(t, T, T_i) = \frac{B(t,T) - B(t,T_i)}{C_{i,N}(t)}, \quad (18)$$

$$P_t^i := \sum_{j=1}^{i} \frac{\delta_j B(t,T_j)}{B(t,T)} \quad (19)$$

and

$$\Pi_t^i := \prod_{j=1}^{i} \left(1 + \delta_j S_j(t)\right).$$

Notice that P_t^i in (19) should become a local martingale under the forward measure $\mathbb{P}_T$. Analogous to (5), it follows from (18) and (19) that, for $i = 1, \ldots, N$

$$\left(1 + \delta_i S_i(t)\right) P_t^i = \delta_i + P_t^{i-1}, \quad (20)$$

and, by multiplying both sides with Π_t^{i-1},

$$\Pi_t^i P_t^i = \delta_i \Pi_t^{i-1} + \Pi_t^{i-1} P_t^{i-1},$$

where $P_t^0 \equiv 0$ and $\Pi_t^0 \equiv 1$. Using induction starting at $i = 1$, it follows that

$$P_t^i = \frac{\sum_{j=1}^{i} \delta_j \Pi_t^{j-1}}{\Pi_t^i} = \sum_{j=1}^{i} \frac{\delta_j}{\prod_{k=j}^{i}(1 + \delta_k S_k(t))}. \tag{21}$$

The model we want to study has a form similar to that of (7), which means that under the measure $\mathbb{P}_T$ the forward swap rates S_i, $i = 1, \ldots, N$ have the dynamics specified as in (22).

Theorem 2.2 *Assume that for each $i = 1, \ldots, N$, $\theta_i(\cdot)$ is a bounded $\mathbb{R}^d$-valued function and $G_i : \mathbb{R}_+ \times \mathbb{R}^r \to (-1, \infty)$ is a deterministic function in $G_{\mathrm{loc}}(\mu)$. Let W^T be a standard Brownian motion in $\mathbb{R}^d$ with respect to the measure $\mathbb{P}_T$ and let $\nu^T(dt, dx) = \lambda^T(t, dx)dt$ be the $\mathbb{P}_T$-compensator of μ. Then the model*

$$\frac{dS_i(t)}{S_i(t-)} = \alpha_i(t)dt + \theta_i(t)dW_t^T + \int_{\mathbb{R}^r} G_i(t, x)(\mu - \nu^T)(dt, dx), \tag{22}$$

is arbitrage-free if

$$\begin{aligned}\alpha_i(t) = & \sum_{j=1}^{i} \frac{\Pi_{t-}^j P_{t-}^j}{\Pi_{t-}^i P_{t-}^i} \left(\frac{\delta_j S_j(t-)}{1 + \delta_j S_j(t-)} \right) \theta_j(t)\theta_i(t) \\ & - \int_{\mathbb{R}^r} \frac{G_i(t, x)}{P_{t-}^i} \left[\sum_{j=1}^{i} \frac{\delta_j}{\prod_{k=j}^{i} \left(1 + \delta_k S_k(t-)(1 + G_k(t, x))\right)} - P_{t-}^i \right] \lambda^T(t, dx).\end{aligned} \tag{23}$$

Proof. As we want the model (22) to be arbitrage-free, each of the P_t^i, $i = 1, \ldots, N$, has to be a local martingale under the measure $\mathbb{P}_T$. Applying Itô's product rule to (20) yields

$$dP_t^{i-1} = (1 + \delta_i S_i(t-))\, dP_t^i + \delta_i P_{t-}^i dS_i(t) + \delta_i d\left[S_i, P^i\right]_t. \tag{24}$$

By eliminating the finite variation terms in (24) we obtain

$$\begin{aligned} dP_t^{i-1} = & (1 + \delta_i S_i(t-))\, dP_t^i + P_{t-}^i \delta_i S_i(t-)\theta_i(t) dW_t^T \\ & + \delta_i P_{t-}^i S_i(t-) \int_{\mathbb{R}^r} G_i(t, x)(\mu - \nu^T)(dt, dx) + \delta_i S_i(t-) \int_{\mathbb{R}^r} G_i(t, x) \times \\ & \times \left[\sum_{j=1}^{i} \frac{\delta_j}{\prod_{k=j}^{i} \left(1 + \delta_k S_k(t-)(1 + G_k(t, x))\right)} - P_{t-}^i \right] (\mu - \nu^T)(dt, dx) \\ = & (1 + \delta_i S_i(t-))\, dP_t^i + P_{t-}^i \delta_i S_i(t-)\theta_i(t) dW_t^T + \delta_i S_i(t-) \int_{\mathbb{R}^r} G_i(t, x) \times \\ & \times \left[\sum_{j=1}^{i} \frac{\delta_j}{\prod_{k=j}^{i} \left(1 + \delta_k S_k(t-)(1 + G_k(t, x))\right)} \right] (\mu - \nu^T)(dt, dx). \end{aligned} \tag{25}$$

Multiplying (25) by Π^{i-1}_{t-} we obtain by induction

$$
\begin{aligned}
dP^i_t = &- P^i_{t-} \sum_{j=1}^{i} \frac{\Pi^j_{t-} P^j_{t-}}{\Pi^i_{t-} P^i_{t-}} \left(\frac{\delta_j S_j(t-)}{1+\delta_j S_j(t-)} \right) \theta_j(t) dW^T_t \\
&- P^i_{t-} \sum_{j=1}^{i} \frac{\Pi^j_{t-}}{\Pi^i_{t-} P^i_{t-}} \left(\frac{\delta_j S_j(t-)}{1+\delta_j S_j(t-)} \right) \times \\
&\times \int_{\mathbb{R}^r} G_j(t,x) \left[\sum_{l=1}^{j} \frac{\delta_j}{\prod_{k=l}^{j} \left(1+\delta_l S_k(t-)(1+G_k(t,x))\right)} \right] (\mu - \nu^T)(dt,dx).
\end{aligned}
$$

In order the model (22) to be arbitrage-free, we have to set the finite variation terms in (24) equal to 0 which yields

$$
\begin{aligned}
&\delta_i P^i_{t-} \alpha_i(t) S_i(t-) dt + \delta_i d\langle S^c_i, P^{i,c} \rangle_t + \\
&+ \delta_i S_i(t-) \int_{\mathbb{R}^r} G_i(t,x) \left[\sum_{j=1}^{i} \frac{\delta_j}{\prod_{k=j}^{i} \left(1+\delta_k S_k(t-)(1+G_k(t,x))\right)} - P^i_{t-} \right] \nu^T(dt,dx) = 0,
\end{aligned}
$$

from where the condition (23) follows.

□

3 The Lévy swap rate model

In Section 2 we discussed a general semimartingale framework for swap rates under the terminal measure. In this section we model the swap rates under the corresponding forward swap measures. In this sense we extend the backward modelling approach of Rutkowski (1999, 2001) where the swap rates are driven by a Wiener process under each forward swap measure. We consider again the family of forward swap rates $S(t,T_i) := S(t,T_i,T_N)$ for $i = 0,\ldots,N-1$, which have the same expiry date, but differ in length of the underlying swap agreement. The essence of this approach is that by fixing T_N, one constructs the model backwards in terms of maturities (thus starting from the largest maturity), specifying at each step the change of measure such that the following swap rate process is a local martingale. This type of backward modelling was also used by Eberlein and Özkan (2005) in the context of the forward Lévy Libor model.

We assume that the discrete tenor structure $0 < T_0 < T_1 < \ldots < T_N$ is given, and $\delta_i = T_i - T_{i-1}$ for $i = 1,\ldots,N$. Since we will proceed by backward induction we set $T^*_i = T_{N-i}$ and, in particular, $T^* := T^*_0 = T_N$. Thus, we consider a "reversed" tenor structure $0 < T^*_N < T^*_{N-1} < \ldots < T^*_1 < T^*_0 = T_N$. For the explicit construction of the forward swap rates we make the following assumptions:

Assumption 3.1
For any maturity T_i, $i = 0,\ldots,N-1$, there exists a bounded and continuous deterministic function $\gamma(\cdot,T_i) \geq 0$, which represents the volatility of the forward swap rate process $S(.,T_i)$. The bound for the volatilities will be specified later.

Assumption 3.2
We assume that the initial term structure of interest rates, specified by bond prices $B(0,T_i)$, $i =$

$0, \ldots, N$, *is given and that* $B(0, T_i)$ *are strictly decreasing in the second variable, i.e.* $B(0, T_i) > B(0, T_{i+1})$, $i = 0, \ldots, N-1$. *Consequently, the initial term structure* $S(0, T_i, T_N)$ *of forward swap rates is given by*

$$S(0, T_i, T_N) = \frac{B(0, T_i) - B(0, T_N)}{C_{i,N}(0)}.$$

We assume a complete stochastic basis $(\Omega, \mathcal{F}_{T^*}, \mathbb{P}_{T^*}, (\mathcal{F}_t)_{0<t\leq T^*})$ to be given. Suppose again that a family of bond prices $B(t, T_m)$, $m = 1, \ldots, N$, is given. For any $m = 1, \ldots, N-1$ the accrual factor (1) can be rewritten as

$$C_{N-m,N}(t) = \sum_{l=N-m+1}^{N} \delta_l B(t, T_l) = \sum_{k=0}^{m-1} \delta_{N-k} B(t, T_k^*), \quad (t \in [0, T_{N-m+1}]) \,. \tag{26}$$

We also define the relative bond prices for a fixed $i = 0, \ldots, N$, and for every $k = 0, \ldots, N$, by

$$Z_{N-i+1}(t, T_k) := \frac{B(t, T_k)}{C_{i-1,N}(t)} = \frac{B(t, T_k)}{\delta_i B(t, T_i) + \cdots + \delta_N B(t, T_N)}, \tag{27}$$

which can be rewritten in terms of backward dates as

$$Z_m(t, T_k^*) = \frac{B(t, T_k^*)}{C_{N-m,N}(t)} = \frac{B(t, T_k^*)}{\delta_{N-m+1} B(t, T_{m-1}^*) + \ldots + \delta_N B(t, T^*)}, \quad t \in [0, T_k^* \wedge T_{m-1}^*], \tag{28}$$

for any fixed $m \in \{1, \ldots, N\}$. For all $t \in [0, T_m^*]$ the forward swap rate for date T_m^* equals

$$\begin{aligned} S(t, T_m^*) &= \frac{B(t, T_m^*) - B(t, T^*)}{\delta_{N-m+1} B(t, T_{m-1}^*) + \ldots + \delta_N B(t, T^*)} \\ &= Z_m(t, T_m^*) - Z_m(t, T^*). \end{aligned} \tag{29}$$

Remark 3.1 *Since obviously* $C_{N-1,N}(t) = \delta_N B(t, T_N) = \delta_N B(t, T^*)$, *it is evident that*

$$Z_1(t, T_k^*) = \frac{B(t, T_k^*)}{C_{N-1,N}(t)} = \frac{B(t, T_k^*)}{\delta_N B(t, T^*)} = \frac{1}{\delta_N} F_B(t, T_k^*, T^*), \tag{30}$$

where $F_B(t, T_k^*, T^*)$ *stands for the forward process corresponding to the time points* T_k^* *and* T^*. *The related forward martingale measure is assumed to be* $\mathbb{P}_{T^*}$ *as given above.*

We postulate that the forward swap measure for the date T^*, which we now denote by $\widetilde{\mathbb{P}}_{T^*}$, coincides with $\mathbb{P}_{T^*}$. Note that in order to stress the fact that we are dealing with the forward swap measure and to avoid any confusion with a forward measure, we add the tilde.

We proceed with the backward construction of forward swap measures. We start by defining the forward swap rate for the date T_1^* by postulating that $S(\cdot, T_1^*)$ is given by

$$S(t, T_1^*) = S(0, T_1^*) \exp\left(\int_0^t \gamma(s, T_1^*) d\widetilde{L}_s^{T^*} \right), \tag{31}$$

where

$$\widetilde{L}_t^{T^*} = \int_0^t b_s ds + \int_0^t \sqrt{c_s} \widetilde{W}_s^{T^*} + \int_0^t \int_{\mathbb{R}} x \left(\mu^L - \widetilde{\nu}^{T^*,L} \right) (ds, dx) \tag{32}$$

is a non-homogeneous Lévy process under $\widetilde{\mathbb{P}}_{T^*}$, i.e. $\widetilde{W}^{T^*}$ is a standard Brownian motion, c_s is positive such that $\int_0^{T^*} (\mid b_s \mid + c_s)\, ds < \infty$, μ^L is the random measure of jumps of the process, and $\widetilde{\nu}^{T^*,L}(ds, dx) = F_s(dx)ds$ is the $\widetilde{P}_{T^*}$-compensator of μ^L. We assume that the Lévy measures F_s, which are measures on $\mathbb{R}$ with $F_s(\{0\}) = 0$ and $\int_0^{T^*}\int_{\mathbb{R}} \left(x^2 \wedge 1\right) F_s(dx)ds < \infty$, satisfy the following additional integrability condition

$$\int_0^{T^*}\int_{|x|>1} \exp(ux)F_s(dx)ds < \infty,$$

for $\mid u \mid \leq (1+\varepsilon)M$, where M, ε are positive constants such that $\sum_{i=1}^n \gamma(\cdot, T_i^*) \leq M$. For the sake of notation we consider a 1-dimensional driving process $(\widetilde{L}_t^{T^*})$ only. The extension to a higher dimensional process is straightforward.

The swap rate process $S(t, T_1^*)$ has to be a martingale under $\widetilde{\mathbb{P}}_{T^*}$. This is achieved through the specification of (b_s). Therefore, we choose the drift characteristics (b_s) such that

$$\int_0^t \gamma(s, T_1^*) b_s ds = -\left(\int_0^t\int_{\mathbb{R}} \left(\mathrm{e}^{\gamma(s,T_1^*)x} - 1 - \gamma(s, T_1^*)x\right)\widetilde{\nu}^{T^*,L}(ds, dx) + \frac{1}{2}\int_0^t c_s \gamma^2(s, T_1^*)ds\right). \tag{33}$$

With this specification $S(t, T_1^*)$ can be written as a stochastic exponential $S(t, T_1^*) = S(0, T_1^*)\mathcal{E}_t\left(V(\cdot, T_1^*)\right)$ where

$$V(t, T_1^*) = \int_0^t \gamma(s, T_1^*)\sqrt{c_s} d\widetilde{W}_s^{T^*} + \int_0^t\int_{\mathbb{R}} \left(\mathrm{e}^{\gamma(s,T_1^*)x} - 1\right)\left(\mu^L - \widetilde{\nu}^{T^*,L}\right)(ds, dx).$$

The $\widetilde{\mathbb{P}}_{T^*}$-dynamics of the forward swap rate is then given by

$$dS(t, T_1^*) = S(t-, T_1^*)\left(\gamma(t, T_1^*)\sqrt{c_t} d\widetilde{W}_t^{T^*} + \int_{\mathbb{R}} \left(\mathrm{e}^{\gamma(t,T_1^*)x} - 1\right)\left(\mu^L - \widetilde{\nu}^{T^*,L}\right)(dt, dx)\right), \tag{34}$$

for all $t \in [0, T_1^*]$, where $\widetilde{W}^{T^*} = W^{T^*}$ and $\widetilde{\nu}^{T^*,L} = \nu^{T^*,L}$, with initial condition

$$S(0, T_1^*) = \frac{B(0, T_1^*) - B(0, T^*)}{\delta_N B(0, T^*)}.$$

Now we need to specify the process $S(\cdot, T_2^*)$, and the martingale measure for the date T_1^*. Referring to Remark 3.1 we have that $\widetilde{\mathbb{P}}_{T^*} = \mathbb{P}_{T^*}$, and

$$Z_1(\cdot, T_k^*) = \frac{1}{\delta_N}\frac{B(\cdot, T_k^*)}{B(\cdot, T^*)} = \frac{1}{\delta_N} F_B(\cdot, T_k^*, T^*)$$

follows a (strictly) positive local martingale under $\widetilde{\mathbb{P}}_{T^*}$. The dynamics of Z_1 can be expressed in a general form as

$$dZ_1(t,T_k^*) = Z_1(t-,T_k^*)\left(\alpha_1(t,T_k^*)\,d\widetilde{W}_t^{T^*} + \int_{\mathbb{R}} \beta_1(x,t,T_k^*)\left(\mu^L - \widetilde{\nu}^{T^*,L}\right)(dt,dx)\right) \tag{35}$$

for some α_1 and β_1 (index 1 refers to the corresponding index in Z_1), and for any date T_k^*. Consider now the relative bond price $Z_2(\cdot,T_k^*)$,

$$Z_2(t,T_k^*) = \frac{B(t,T_k^*)}{\delta_N B(t,T^*) + \delta_{N-1}B(t,T_1^*)} = \frac{Z_1(t,T_k^*)}{\delta_{N-1}Z_1(t,T_1^*)+1}. \tag{36}$$

We further use Lemma A.1 from the Appendix. This lemma corresponds to Lemma 2.3 in Rutkowski (2001), which we generalized to our setting. We modified it slightly to include the measure transform. Based on the dynamics (35), and by applying the lemma to the processes $G = Z_1(\cdot,T_k^*)$ and $H = \delta_{N-1}Z_1(\cdot,T_1^*)$, we define the forward swap measure associated with the date T_1^*, by setting its Radon-Nikodym density as the stochastic exponential at time T_1^*,

$$\frac{d\widetilde{\mathbb{P}}_{T_1^*}}{d\widetilde{\mathbb{P}}_{T^*}} = \mathcal{E}_{T_1^*}(M^1),$$

where

$$\begin{aligned} M_t^1 = &\int_0^t \frac{\delta_{N-1}Z_1(s-,T_1^*)\alpha_1(s,T_1^*)}{1+\delta_{N-1}Z_1(s-,T_1^*)} d\widetilde{W}_s^{T^*} \\ &+ \int_0^t \int_{\mathbb{R}} \left(\frac{\delta_{N-1}Z_1(s-,T_1^*)\beta_1(x,t,T_1^*)}{1+\delta_{N-1}Z_1(s-,T_1^*)}\right)\left(\mu^L - \widetilde{\nu}^{T^*,L}\right)(ds,dx). \end{aligned}$$

Then

$$\widetilde{W}_t^{T_1^*} = \widetilde{W}_t^{T^*} - \int_0^t \frac{\delta_{N-1}Z_1(s-,T_1^*)\alpha_1(s,T_1^*)}{1+\delta_{N-1}Z_1(s-,T_1^*)} ds$$

is the forward Brownian motion for the date T_1^* and

$$\widetilde{\nu}^{T_1^*,L} = \left(1 + \frac{\delta_{N-1}Z_1(s-,T_1^*)\beta_1(x,s,T_1^*)}{1+\delta_{N-1}Z_1(s-,T_1^*)}\right)\widetilde{\nu}^{T^*,L}$$

is the $\widetilde{\mathbb{P}}_{T_1^*}$-compensator of μ^L. According to Lemma A.1, the process $Z_2(\cdot,T_k^*)$ is a local martingale under the new measure $\widetilde{\mathbb{P}}_{T_1^*}$. In order to express the measure change through the swap rates instead of using the relative bond prices, we use the following relations. First notice that from (29) and (30) we obtain that

$$Z_1(t,T_1^*) = \frac{B(t,T_1^*)}{\delta_N B(t,T^*)} = S(t,T_1^*) + Z_1(t,T^*) = S(t,T_1^*) + \delta_N^{-1}. \tag{37}$$

Differentiating both sides of the last equality and invoking (34) and (35), we obtain

$$\begin{aligned} &Z_1(t-,T_1^*)\alpha_1(t,T_1^*)\,d\widetilde{W}_t^{T^*} + Z_1(t-,T_1^*)\int_{\mathbb{R}}\beta_1(x,t,T_1^*)\left(\mu^L - \widetilde{\nu}^{T^*,L}\right)(dt,dx) \\ &= S(t-,T_1^*)\gamma_1(t,T_1^*)d\widetilde{W}_t^{T^*} + S(t-,T_1^*)\int_{\mathbb{R}}\gamma_2(x,t,T_1^*)(\mu^L - \widetilde{\nu}^{T^*,L})(dt,dx), \end{aligned} \tag{38}$$

where for the shorthand notation, we have set $\gamma_1(t,T_1^*) = \gamma(t,T_1^*)\sqrt{c_t}$ and $\gamma_2(x,t,T_1^*) = e^{\gamma(t,T_1^*)x} - 1$. As the Gaussian and the jump part of a semimartingale do not interact (see Jacod and Shiryaev (1987, II.2.34)), in order for this equality to hold we set

$$Z_1(t-,T_1^*)\alpha_1(t,T_1^*) = S(t-,T_1^*)\gamma_1(t,T_1^*)$$

$$Z_1(t-,T_1^*)\beta_1(x,t,T_1^*) = S(t-,T_1^*)\gamma_2(x,t,T_1^*).$$

Consequently, $\widetilde{W}^{T_1^*}$ is explicitly given by the formula

$$\widetilde{W}_t^{T_1^*} = \widetilde{W}_t^{T^*} - \int_0^t \frac{\delta_{N-1}S(s-,T_1^*)\gamma_1(s,T_1^*)}{1+\delta_{N-1}\delta_N^{-1}+\delta_{N-1}S(s-,T_1^*)}ds,$$

and the $\widetilde{\mathbb{P}}_{T_1^*}$-compensator of μ^L by

$$\widetilde{\nu}^{T_1^*,L} = \left(1+\frac{\delta_{N-1}S(s-,T_1^*)\gamma_2(x,s,T_1^*)}{1+\delta_{N-1}\delta_N^{-1}+\delta_{N-1}S(s-,T_1^*)}\right)\widetilde{\nu}^{T^*,L}.$$

Now we can define the next forward swap rate $S(t,T_2^*)$ by postulating that under $\widetilde{\mathbb{P}}_{T_1^*}$

$$S(t,T_2^*) = S(0,T_2^*)\exp\left(\int_0^t \gamma(s,T_2^*)d\widetilde{L}_s^{T_1^*}\right), \tag{39}$$

where

$$\widetilde{L}_t^{T_1^*} = \int_0^t b_s^{T_1^*}ds + \int_0^t \sqrt{c_s}\widetilde{W}_s^{T_1^*} + \int_0^t\int_{\mathbb{R}} x\left(\mu^L - \widetilde{\nu}^{T_1^*,L}\right)(ds,dx). \tag{40}$$

In order to make $S(t,T_2^*)$ a $\widetilde{P}_{T_1^*}$-martingale we choose the drift term $(b_s^{T_1^*})$ such that

$$\int_0^t \gamma(s,T_2^*)b_s^{T_1^*}ds = -\left(\int_0^t\int_{\mathbb{R}}\left(\mathrm{e}^{\gamma(s,T_2^*)x} - 1 - \gamma(s,T_2^*)x\right)\widetilde{\nu}^{T_1^*,L}(ds,dx) + \frac{1}{2}\int_0^t c_s\gamma^2(s,T_2^*)ds\right). \tag{41}$$

With this specification of the drift term, $S(t,T_2^*)$ can be written as a stochastic exponential $S(t,T_2^*) = S(0,T_2^*)\mathcal{E}_t\left(V(\cdot,T_2^*)\right)$ where

$$V(t,T_2^*) = \int_0^t \gamma(s,T_2^*)\sqrt{c_s}d\widetilde{W}_s^{T_1^*} + \int_0^t\int_{\mathbb{R}}\left(\mathrm{e}^{\gamma(s,T_2^*)x}-1\right)\left(\mu^L - \widetilde{\nu}^{T_1^*,L}\right)(ds,dx).$$

The $\widetilde{\mathbb{P}}_{T_1^*}$-dynamics of the forward swap rate is then given by

$$dS(t,T_2^*) = S(t-,T_2^*)\left(\gamma(t,T_2^*)\sqrt{c_t}d\widetilde{W}_t^{T_1^*} + \int_{\mathbb{R}}\left(\mathrm{e}^{\gamma(t,T_2^*)x}-1\right)\left(\mu^L - \widetilde{\nu}^{T_1^*,L}\right)(dt,dx)\right) \tag{42}$$

for all $t \in [0, T_2^*]$ with the initial condition

$$S(0, T_2^*) = \frac{B(0, T_2^*) - B(0, T^*)}{\delta_{N-1} B(0, T_1^*) + \delta_N B(0, T^*)}.$$

As the next inductive step, we need to specify the process $S(t, T_3^*)$ and the martingale measure for the date T_2^*. From the previous step we know that the process $Z_2(\cdot, T_k^*)$ follows a (strictly) positive local martingale under $\widetilde{\mathbb{P}}_{T_1^*}$. In general, the dynamics of Z_2 can then be expressed as

$$dZ_2(t, T_k^*) = Z_2(t-, T_k^*) \left(\alpha_2(t, T_k^*)\, d\widetilde{W}_t^{T_1^*} + \int_{\mathbb{R}} \beta_2(x, t, T_k^*) \left(\mu^L - \widetilde{\nu}^{T_1^*, L} \right) (dt, dx) \right) \qquad (43)$$

for some α_2 and β_2 (index 2 now refers to the corresponding index in Z_2), and for any date T_k^*. At this step we consider the relative bond prices $Z_3(\cdot, T_k^*)$,

$$Z_3(t, T_k^*) = \frac{B(t, T_k^*)}{\delta_{N-2} B(t, T_2^*) + \delta_{N-1} B(t, T_1^*) + \delta_N B(t, T^*)} = \frac{Z_2(t, T_k^*)}{1 + \delta_{N-2} Z_2(t, T_2^*)}. \qquad (44)$$

Applying Lemma A.1 to processes $G = Z_2(\cdot, T_k^*)$ and $H = \delta_{N-2} Z_2(\cdot, T_2^*)$, we define the forward swap measure associated with date T_2^*, by setting its Radon-Nikodym density as the stochastic exponential at time T_2^*,

$$\frac{d\widetilde{\mathbb{P}}_{T_2^*}}{d\widetilde{\mathbb{P}}_{T_1^*}} = \mathcal{E}_{T_2^*}(M^2),$$

where

$$M_t^2 = \int_0^t \frac{\delta_{N-2} Z_2(s-, T_2^*) \alpha_2(s, T_2^*)}{1 + \delta_{N-2} Z_2(s-, T_2^*)} d\widetilde{W}_s^{T_1^*} + \int_0^t \int_{\mathbb{R}} \left(\frac{\delta_{N-2} Z_2(s-, T_2^*) \beta_2(x, s, T_2^*)}{1 + \delta_{N-2} Z_2(s-, T_2^*)} \right) \left(\mu^L - \widetilde{\nu}^{T_1^*, L} \right) (ds, dx).$$

Then

$$\widetilde{W}_t^{T_2^*} = \widetilde{W}_t^{T_1^*} - \int_0^t \frac{\delta_{N-2} Z_2(s-, T_2^*) \alpha_2(s, T_2^*)}{1 + \delta_{N-2} Z_2(s-, T_2^*)}\, ds \qquad (t \in [0, T_2^*])$$

is the forward Brownian motion for the date T_2^* and

$$\widetilde{\nu}^{T_2^*, L} = \left(1 + \frac{\delta_{N-2} Z_2(s-, T_2^*) \beta_2(x, s, T_2^*)}{1 + \delta_{N-2} Z_2(s-, T_2^*)} \right) \widetilde{\nu}^{T_1^*, L}.$$

is the $\widetilde{\mathbb{P}}_{T_2^*}$-compensator of μ^L. According to Lemma A.1, the process $Z_3(\cdot, T_k^*)$ is a local martingale under the new measure $\widetilde{\mathbb{P}}_{T_2^*}$. In order to express the measure change through the swap rates instead of using the relative bond prices, notice that

$$
\begin{aligned}
Z_2(t, T_2^*) &\overset{(28)}{=} \frac{B(t, T_2^*)}{\delta_{N-1} B(t, T_1^*) + \delta_N B(t, T^*)} \\
&\overset{(29)}{=} S(t, T_2^*) + Z_2(t, T^*) \\
&\overset{(36)}{\overset{(37)}{=}} S(t, T_2^*) + \frac{Z_1(t, T^*)}{1 + \delta_{N-1} Z_1(t, T^*) + \delta_{N-1} S(t, T_1^*)},
\end{aligned}
$$

where the process $Z_1(\cdot, T^*)$ is already known from the previous step, cfr. (37). Differentiating the last equality we can thus find the coefficients of the Gaussian and the discontinuous terms of the process $Z_2(\cdot, T^*)$, and consequently define $\widetilde{\mathbb{P}}_{T_2^*}$.

Extension to the general case is straightforward. Consider the induction step with respect to m. Suppose that we have already defined the forward swap rates $S(t, T_1^*), \ldots, S(t, T_m^*)$, and specified the forward swap measure $\widetilde{\mathbb{P}}_{T_{m-1}^*}$. At this step we would like to determine the forward swap measure $\widetilde{\mathbb{P}}_{T_m^*}$, and the forward swap rate $S(\cdot, T_{m+1}^*)$. We consider the relative bond prices

$$
Z_{m+1}(t, T_k^*) = \frac{B(t, T_k^*)}{C_{N-(m+1),N}(t)} = \frac{B(t, T_k^*)}{\delta_{N-m} B(t, T_m^*) + \cdots + \delta_N B(t, T^*)} = \frac{Z_m(t, T_k^*)}{1 + \delta_{N-m} Z_m(t, T_m^*)}.
$$

Applying Lemma A.1 to processes $G = Z_m(\cdot, T_k^*)$ and $H = \delta_{N-m} Z_m(\cdot, T_m^*)$, it is clear that we can define the forward swap measure associated with date T_m^*, by setting its Radon-Nikodym density as

$$
\frac{d\widetilde{\mathbb{P}}_{T_m^*}}{d\widetilde{\mathbb{P}}_{T_{m-1}^*}} = \mathcal{E}_{T_m^*}(M^m), \tag{45}
$$

where

$$
\begin{aligned}
M_t^m = &\int_0^t \frac{\delta_{N-m} Z_m(s-, T_m^*) \alpha_m(s, T_m^*)}{1 + \delta_{N-m} Z_m(s-, T_m^*)} d\widetilde{W}_s^{T_{m-1}^*} \\
&+ \int_0^t \int_{\mathbb{R}} \left(\frac{\delta_{N-m} Z_m(s-, T_m^*) \beta_m(x, s, T_m^*)}{1 + \delta_{N-m} Z_m(s-, T_m^*)} \right) \left(\mu^L - \widetilde{\nu}^{T_{m-1}^*, L} \right)(ds, dx).
\end{aligned} \tag{46}
$$

Then

$$
\widetilde{W}_t^{T_m^*} = \widetilde{W}_t^{T_{m-1}^*} - \int_0^t \frac{\delta_{N-m} Z_m(s-, T_m^*) \alpha_m(s, T_m^*)}{1 + \delta_{N-m} Z_m(s-, T_m^*)} \, ds \qquad (t \in [0, T_m^*])
$$

is the forward Brownian motion for the date T_m^* and

$$
\widetilde{\nu}^{T_m^*, L} = \left(1 + \frac{\delta_{N-m} Z_m(s-, T_m^*) \beta_m(x, s, T_m^*)}{1 + \delta_{N-m} Z_m(s-, T_m^*)} \right) \widetilde{\nu}^{T_{m-1}^*, L}
$$

is the $\widetilde{\mathbb{P}}_{T_m^*}$-compensator of μ^L. Therefore it suffices to analyze the process

$$
Z_m(t, T_m^*) = \frac{B(t, T_m^*)}{\delta_{N-m+1} B(t, T_{m-1}^*) + \cdots + \delta_N B(t, T^*)} = S(t, T_m^*) + Z_m(t, T^*),
$$

where

$$Z_m(t,T^*) = \frac{Z_{m-1}(t,T^*)}{1+\delta_{N-m+1}Z_{m-1}(t,T^*)+\delta_{N-m+1}S(t,T^*_{m-1})}.$$

The process $Z_{m-1}(\cdot,T^*)$ is known from the preceding step and is a function of the forward swap rates $S(t,T^*_1),\ldots,S(t,T^*_{m-1})$. Consequently, we can express $Z_m(\cdot,T^*)$ using the terms $S(t,T^*_1),\ldots,S(t,T^*_{m-1})$ and their volatilities. Having found the probability measure $\widetilde{\mathbb{P}}_{T^*_m}$, we introduce the forward swap rate $S(t,T^*_{m+1})$,

$$S(t,T^*_{m+1}) = S(0,T^*_{m+1})\exp\left(\int_0^t \gamma(s,T^*_{m+1})d\widetilde{L}^{T^*_m}_s\right),$$

and so forth.

The precise form of $\frac{d\widetilde{\mathbb{P}}_{T^*_m}}{d\widetilde{\mathbb{P}}_{T^*_{m-1}}} \equiv \frac{d\widetilde{\mathbb{P}}_{T_i}}{d\widetilde{\mathbb{P}}_{T_{i+1}}}$ in (45)-(46) is easy to find by using the following reasoning. Recall that $\widetilde{\mathbb{P}}_i$ is the probability measure associated to the numéraire $C_{i-1,N}$ and $\mathbb{P}_N$ is the probability measure associated to the numéraire $B(\cdot,T_N)$. According to the change of numéraire theorem (see Geman, El Karoui and Rochet (1995)) we set

$$\rho^i_t := \frac{d\widetilde{\mathbb{P}}_i}{d\mathbb{P}_N}|_{\mathcal{F}_t} = \frac{C_{i-1,N}(t)}{B(t,T_N)}\cdot\frac{B(0,T_N)}{C_{i-1,N}(0)} = \frac{P^{i-1}_t}{P^{i-1}_0}.$$

In view of Girsanov's theorem and equation (14) we can deduce the Radon-Nikodym derivative ρ^i_t as a stochastic exponential given by

$$\frac{d\widetilde{\mathbb{P}}_i}{d\mathbb{P}_N}|_{\mathcal{F}_t} = \mathcal{E}_t\left(\int_0^\cdot \varphi^i_t dW^N_t + \int_0^\cdot\int_{\mathbb{R}}\left(Y^i(t,x)-1\right)\left(\mu^L-\nu^N\right)(dt,dx)\right) =: \mathcal{E}_t\left(X^i\right)$$

where

$$\varphi^i_t = \sum_{j=i}^{N-1}\frac{\Psi^{j-1}_{t-}P^j_{t-}}{\Psi^{i-2}_{t-}P^{i-1}_{t-}}\left(\frac{\delta_j S_j(t-)}{1+\delta_j S_j(t-)}\right)\gamma_1(t,T_j) \tag{47}$$

and

$$\begin{aligned}Y^i(t,x)-1 &= \sum_{j=i}^{N-1}\frac{\Psi^{j-1}_{t-}}{\Psi^{i-2}_{t-}P^{i-1}_{t-}}\left(\frac{\delta_j S_j(t-)}{1+\delta_j S_j(t-)}\right)\gamma_2(x,t,T_j)\times\\ &\quad\times\left[\sum_{l=j}^{N-1}\delta_{l+1}\prod_{k=j+1}^{l}\left(1+\delta_k S_k(t-)(1+\gamma_2(x,t,T_k))\right)\right].\end{aligned} \tag{48}$$

We proceed by writing

$$\frac{d\widetilde{\mathbb{P}}_i}{d\widetilde{\mathbb{P}}_k} = \frac{d\widetilde{\mathbb{P}}_i}{d\mathbb{P}_N}\frac{d\mathbb{P}_N}{d\widetilde{\mathbb{P}}_k} = \frac{d\widetilde{\mathbb{P}}_i}{d\mathbb{P}_N}\frac{1}{\frac{d\widetilde{\mathbb{P}}_k}{d\mathbb{P}_N}} = \mathcal{E}(X^i)\frac{1}{\mathcal{E}(X^k)}.$$

By applying Lemmas 2.4 and 2.6 from Kallsen and Shiryaev (2002) and taking into account that $\mathcal{E}(U)\mathcal{E}(V) = \mathcal{E}(U+V+[U,V])$ for any two semimartingales U and V, we finally obtain the connection between any two forward swap measures $\widetilde{\mathbb{P}}_i$ and $\widetilde{\mathbb{P}}_k$:

$$\frac{d\widetilde{\mathbb{P}}_i}{d\widetilde{\mathbb{P}}_k} = \mathcal{E}(Z), \tag{49}$$

with

$$\begin{aligned} Z_t = & \int_0^t \left(\sum_{j=i}^{N-1} \frac{\xi_{s-}^j \gamma_1(s,T_j)}{\Psi_{s-}^{i-2} P_{s-}^{i-1}} - \sum_{j=k}^{N-1} \frac{\xi_{s-}^j \gamma_1(s,T_j)}{\Psi_{s-}^{k-2} P_{s-}^{k-1}} \right) d\widetilde{W}_s^k \\ & + \int_0^t \int_{\mathbb{R}} \left(\frac{\sum_{j=i}^{N-1} \frac{\zeta_j(s,x)}{\Psi_{s-}^{i-2} P_{s-}^{i-1}} + 1}{\sum_{j=k}^{N-1} \frac{\zeta_j(s,x)}{\Psi_{s-}^{k-2} P_{s-}^{k-1}} + 1} - 1 \right) (\mu^L - \widetilde{\nu}^{k,L})(ds, dx) \end{aligned} \tag{50}$$

where $\xi_{s-}^j = \Psi_{s-}^{j-1} P_{s-}^j \left(\frac{\delta_j S_j(s-)}{1+\delta_j S_j(s-)} \right)$,

$$\zeta_j(s,x) = \Psi_{s-}^{j-1} \left(\frac{\delta_j S_j(s-)}{1+\delta_j S_j(s-)} \right) \gamma_2(x,s,T_j) \left[\sum_{l=j}^{N-1} \delta_{l+1} \prod_{v=j+1}^{l} \big(1 + \delta_v S_v(s-)(1+\gamma_2(x,s,T_v))\big) \right],$$

$\widetilde{W}^k$ is the standard Brownian motion with respect to the forward swap measure $\widetilde{\mathbb{P}}_k$ and $\widetilde{\nu}^{k,L}$ is the $\widetilde{\mathbb{P}}_k$-compensator of μ^L. By using (49)-(50) we easily obtain the connection between the two consecutive forward swap measures $\widetilde{\mathbb{P}}_i$ and $\widetilde{\mathbb{P}}_{i+1}$. Notice that the process driving the most distant forward swap rate $S(t, T_{N-1}, T_N)$ is a non-homogeneous Lévy process. This is however not true for the process driving $S(t, T_i, T_N)$, $i < N-1$, constructed during the backward induction, since the associated compensators $\widetilde{\nu}^{T_{i+1},L}$ contain random terms. It simplifies the numerical implementation if all driving processes are non-homogeneous Lévy processes. This can be achieved by introducing an approximation, where we replace the random terms by their deterministic initial values at time $t = 0$.

4 Pricing of swaptions

In this section we consider the pricing of swap rate based options, or swaptions, more precisely we shall price a swaption where the underlying swap starts at tenor time point T_i and ends at time point T_N. By using general valuation results (see e.g. Musiela and Rutkowski (1997b)), the time $t = 0$ price of the forward payer swaption is given by

$$\begin{aligned} \mathrm{PS}_0 &= \sum_{k=i+1}^{N} \delta_k B(0,T_k) \mathrm{E}^{\widetilde{\mathbb{P}}_{T_{i+1}}} \left[(S(T_i,T_i,T_N) - K)_+ \right] \\ &= C_{i,N}(0) \mathrm{E}^{\widetilde{\mathbb{P}}_{T_{i+1}}} \left[(S(T_i,T_i,T_N) - K)_+ \right]. \end{aligned} \tag{51}$$

Raible (2000) proposed a method for the evaluation of European stock options in a Lévy setting by using bilateral (or, two-sided) Laplace transforms. The approach is based on the observation that the pricing formula for European options can be represented as a convolution. Whence one can use the fact that the bilateral Laplace transform of a convolution is the product of the bilateral Laplace transforms of the factors (the latter transforms are usually known explicitly). Inversion of the bilateral Laplace transform then yields the option prices as a function of the current price of the underlying asset, and can be accomplished through the Fast Fourier Transform algorithm. See also the related work of Carr and Madan (1999), and Lee (2004).

Eberlein and Özkan (2005) showed how caplets can be priced using bilateral Laplace transforms. This method could also be employed for pricing the forward payer swaptions (51) as we shall shortly explain in the following. We do not give all technical assumptions or details accompanying that approach, instead we refer the reader to Raible (2000).

We concentrate on the purely discontinuous case ($c = 0$ in (32)) since we will use generalized hyperbolic Lévy processes in the implementations. We consider the forward swap rate in the form as given in (31):

$$S(t, T_i, T_N) = S(0, T_i, T_N) \exp\left(\int_0^t \gamma(s, T_i) d\widetilde{L}_s^{T_{i+1}}\right),$$

where the condition (33) now simplifies to

$$\int_0^t \gamma(s, T_i) b_s^{T_{i+1}} ds = -\int_0^t \int_{\mathbb{R}} \left(\mathrm{e}^{\gamma(s,T_i)x} - 1 - \gamma(s, T_i)x\right) \widetilde{\nu}^{T_{i+1},L}(ds, dx).$$

We define

$$X_t := \int_0^t \gamma(s, T_i) d\widetilde{L}_s^{T_{i+1}}$$

so that

$$\begin{aligned} X_{T_i} &= \ln\left(\frac{S(T_i, T_i, T_N)}{S(0, T_i, T_N)}\right) \\ &= \int_0^{T_i} \gamma(s, T_i) b_s^{T_{i+1}} ds + \int_0^{T_i} \int_{\mathbb{R}} x\gamma(s, T_i) \left(\mu^L - \widetilde{\nu}^{T_{i+1},L}\right)(ds, dx). \end{aligned}$$

By defining $w(x, K) := (x - K)_+$, the payoff of the swaption is given by $w(S(T_i, T_i, T_N), K)$ and its price at time $t = 0$ by $C_{i,N}(0)\mathrm{E}^{\widetilde{\mathbb{P}}_{T_{i+1}}}[w(S(T_i, T_i, T_N), K)]$. We consider the modified payoff $\widetilde{w}(x, K) := w(\mathrm{e}^{-x}, K)$. Let $\zeta_i := -\ln S(0, T_i, T_N)$, then $S(T_i, T_i, T_N) = \mathrm{e}^{-\zeta_i + X_{T_i}}$. Furthermore, denote by $V(\zeta_i, K)$ the time zero price of the swaption, and let $L[\widetilde{w}]$ be the bilateral Laplace transform of $\widetilde{w}$:

$$L[\widetilde{w}](z) = \int_{-\infty}^{+\infty} \mathrm{e}^{-zx} \widetilde{w}(x) dx, \quad z = R + \mathrm{i}u \in \mathbb{C}, \;\; R, u \in \mathbb{R}.$$

The price of the swaption at time zero can be written (apart from the discount factor) as a convolution of functions $\widetilde{w}(x)$ and $\rho(x)$, taken at the point ζ_i:

$$\begin{aligned} V(\zeta_i, K) &= C_{i,N}(0)\mathrm{E}^{\widetilde{\mathbb{P}}_{T_{i+1}}}\left[w(\mathrm{e}^{-\zeta_i + X_{T_i}}, K)\right] \\ &= C_{i,N}(0)\mathrm{E}^{\widetilde{\mathbb{P}}_{T_{i+1}}}\left[\widetilde{w}(\zeta_i - X_{T_i}, K)\right] \\ &= C_{i,N}(0) \int_{\mathbb{R}} \widetilde{w}(\zeta_i - x, K)\rho(x)(dx), \end{aligned}$$

where ρ is the density of X_{T_i}. As mentioned above, the bilateral Laplace transform of a convolution equals the product of the bilateral Laplace transforms of the factors. Thus, we have that

$$L[V](R + \mathrm{i}u) = C_{i,N}(0) L[\widetilde{w}](R + \mathrm{i}u) \cdot L[\rho](R + \mathrm{i}u), \qquad (u \in \mathbb{R}). \tag{52}$$

By Theorem B.2 in Raible (2000) the bilateral Laplace integral defining $L[V](z)$ converges absolutely, and $\zeta \mapsto V(\zeta, K)$ is a continuous function. Hence, according to Theorem B.3 in Raible (2000) we can invert the bilateral Laplace transform to obtain the swaption price V:

$$\begin{aligned} V(\zeta_i, K) &= \frac{1}{2\pi \mathrm{i}} \int_{R-\mathrm{i}\infty}^{R+\mathrm{i}\infty} \mathrm{e}^{\zeta_i z} L[V](z) dz \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{\zeta_i (R+\mathrm{i}u)} L[V](R+\mathrm{i}u) du \\ &= \frac{\mathrm{e}^{\zeta_i R}}{2\pi} \lim_{\substack{M \to \infty \\ N \to \infty}} \int_{-M}^{N} \mathrm{e}^{\mathrm{i}u\zeta_i} L[V](R+\mathrm{i}u) du. \end{aligned} \tag{53}$$

Note the identity $L[\rho](R + \mathrm{i}u) = \chi(\mathrm{i}R - u)$, where $\chi(\mathrm{i}R - u) := \mathrm{E}^{\widetilde{\mathbb{P}}_{T_{i+1}}} \left[\mathrm{e}^{\mathrm{i}(\mathrm{i}R-u)X_{T_i}} \right]$ is the extended characteristic function of X_{T_i}. By substituting (52) into (53) we finally obtain the swaption pricing formula

$$V(\zeta_i, K) = C_{i,N}(0) \frac{\mathrm{e}^{\zeta_i R}}{2\pi} \lim_{\substack{M \to \infty \\ N \to \infty}} \int_{-M}^{N} \mathrm{e}^{\mathrm{i}u\zeta_i} L[\widetilde{w}](R+\mathrm{i}u) \chi(\mathrm{i}R - u) du. \tag{54}$$

According to Raible (2000) it is sufficient to consider the case where the strike price equals one, since

$$V(\zeta_i, K) = KV(\zeta_i + \ln K, 1).$$

The bilateral Laplace transform $L[\widetilde{w}]$ for $K = 1$ is given by $L[\widetilde{w}](z) = (z(z+1))^{-1}$, if $\mathrm{Re} z < -1$. We remark that this approach can be used also for more complicated payoff functions as long as the payoff depends only on X_{T_i} (for examples, we refer to Raible (2000)).

The characteristic function of X_{T_i} has the following form and simplifies in the purely discontinuous case (see Eberlein and Raible (1999)):

$$\begin{aligned} \chi(u) &:= \mathrm{E}^{\widetilde{\mathbb{P}}_{T_{i+1}}} \left[\mathrm{e}^{\mathrm{i}uX_{T_i}} \right] \\ &= \exp \Bigg(\mathrm{i}u \int_0^{T_i} \gamma(s, T_i) b_s^{T_{i+1}} ds \\ &\qquad\qquad + \int_0^{T_i} \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}u\gamma(s,T_i)x} - 1 - \mathrm{i}u\gamma(s, T_i)x \right) \widetilde{\nu}^{T_{i+1},L}(ds, dx) \Bigg) \\ &= \exp \Bigg(-\mathrm{i}u \int_0^{T_i} \int_{\mathbb{R}} \left(\mathrm{e}^{\gamma(s,T_i)x} - 1 - \gamma(s, T_i)x \right) \widetilde{\nu}^{T_{i+1},L}(ds, dx) \\ &\qquad\qquad + \int_0^{T_i} \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}u\gamma(s,T_i)x} - 1 - \mathrm{i}u\gamma(s, T_i)x \right) \widetilde{\nu}^{T_{i+1},L}(ds, dx) \Bigg) \\ &= \exp \left(\int_0^{T_i} \int_{\mathbb{R}} \left(\mathrm{e}^{\mathrm{i}u\gamma(s,T_i)x} - \mathrm{i}u\mathrm{e}^{\gamma(s,T_i)x} - (1 - \mathrm{i}u) \right) \widetilde{\nu}^{T_{i+1},L}(ds, dx) \right). \end{aligned} \tag{55}$$

The characteristic function (55) can be determined more precisely once the distribution of L_1 is specified (generalized hyperbolic or normal inverse Gaussian for instance). See e.g. Schoutens (2003) for a number of Lévy processes and their characteristic functions. Hence, we can calculate equation (54) numerically in an efficient way.

5 Concluding remarks

In this paper we derived the dynamics of the forward swap rate process in a semimartingale setting in explicit form and introduced a Lévy swap market model where swap rates are modeled as ordinary exponentials. Starting from the most distant swap rate (which is driven by a non-homogeneous Lévy process), we constructed the other swap rates such that they become martingales under the corresponding forward swap measures. We also showed how swaptions can be priced in this framework by using Laplace transform methods. A topic for future research will be to investigate further swap derivatives in the Lévy framework and to test the performance of the model.

A Appendix

In this Appendix we will prove a general result which is applied in Section 3. The lemma corresponds to Lemma 2.3 in Rutkowski (2001), which we generalized to a semimartingale setting. It is also modified slightly to include the measure transform.

Lemma A.1 *Let G, H be real-valued adapted processes under some probability measure $\mathbb{P}$, satisfying the following SDEs*

$$dG_t = g_1(t)\,dW_t + \int_{\mathbb{R}} g_2(t,x)\,(\mu-\nu)(dt,dx) \tag{56}$$

$$dH_t = h_1(t)\,dW_t + \int_{\mathbb{R}} h_2(t,x)\,(\mu-\nu)(dt,dx) \tag{57}$$

where W_t is $\mathbb{P}$-Brownian motion and $\nu(dt,dx)$ is the $\mathbb{P}$-compensator of the random measure of jumps μ. Let g_1, h_1 be square-integrable $\mathbb{P}$-a.s. and $g_2, h_2 \in G_{\mathrm{loc}}(\mu)$. Suppose $H_t > -1$. Define $Y_t := (1+H_t)^{-1}$. Then the process YG has the local martingale dynamics

$$\begin{aligned} d(YG)_t &= Y_{t-}(g_1(t) - Y_{t-}G_{t-}h_1(t))d\widetilde{W}_t \\ &\quad + \int_{\mathbb{R}} \left(\frac{G_{t-} + g_2(t,x)}{1 + H_{t-} + h_2(t,x)} - \frac{G_{t-}}{1+H_{t-}} \right) (\mu - \widetilde{\nu})(dt,dx) \end{aligned}$$

under a new measure $\widetilde{\mathbb{P}}$, $\widetilde{\mathbb{P}} \overset{\mathrm{loc}}{\ll} \mathbb{P}$, where $\widetilde{W}_t$ is a $\widetilde{\mathbb{P}}$-Brownian motion, $d\widetilde{W}_t = dW_t - Y_{t-}h_1(t)\,dt$, and $\widetilde{\nu}(dt,dx)$ is the $\widetilde{\mathbb{P}}$-compensator of μ given by

$$\widetilde{\nu}(dt,dx) = (1 + Y_{t-}h_2(t,x))\,\nu(dt,dx).$$

The density process is given by $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}(M)$, where

$$M_t = \int_0^t Y_{s-}h_1(s)dW_s + \int_0^t \int_{\mathbb{R}} Y_{s-}h_2(s,x)(\mu-\nu)(ds,dx).$$

Proof. We use the following short-hand notation in order to improve the readability: $g_1 := g_1(t)$, $h_1 := h_1(t)$, $g_2 := g_2(t,x)$ and $h_2 := h_2(t,x)$. By applying Itô's formula for semimartingales (Jacod and Shiryaev (1987), I.4.57), we obtain

$$d\left(\frac{G_t}{1+H_t}\right) = \frac{1}{1+H_{t-}}dG_t - \frac{G_{t-}}{(1+H_{t-})^2}dH_t$$

$$
\begin{aligned}
&+\frac{1}{2}\left(-\frac{g_1h_1}{(1+H_{t-})^2}dt+\frac{2G_{t-}h_1^2}{(1+H_{t-})^3}dt-\frac{g_1h_1}{(1+H_{t-})^2}dt\right)\\
&+\int_{\mathbb{R}}\left(\frac{G_{t-}+g_2}{1+H_{t-}+h_2}-\frac{G_{t-}}{1+H_{t-}}-\left[\frac{g_2}{1+H_{t-}}-\frac{G_{t-}h_2}{(1+H_{t-})^2}\right]\right)\mu(dt,dx)\\
=&\left(-\frac{g_1h_1}{(1+H_{t-})^2}+\frac{G_{t-}h_1^2}{(1+H_{t-})^3}\right)dt\\
&+\left(\frac{g_1}{1+H_{t-}}-\frac{G_{t-}h_1}{(1+H_{t-})^2}\right)dW_t\\
&+\int_{\mathbb{R}}\frac{g_2}{1+H_{t-}}(\mu-\nu)(dt,dx)-\int_{\mathbb{R}}\frac{G_{t-}h_2}{(1+H_{t-})^2}(\mu-\nu)(dt,dx)\\
&+\int_{\mathbb{R}}\left(\frac{G_{t-}+g_2}{1+H_{t-}+h_2}-\frac{G_{t-}}{1+H_{t-}}-\left[\frac{g_2}{1+H_{t-}}-\frac{G_{t-}h_2}{(1+H_{t-})^2}\right]\right)\mu(dt,dx).
\end{aligned}
\tag{58}
$$

The continuous part of (58) can be dealt with exactly the same way as in Lemma 2.3 in Rutkowski (2001), yielding

$$
\begin{aligned}
&\left(-\frac{g_1h_1}{(1+H_{t-})^2}+\frac{G_{t-}h_1^2}{(1+H_{t-})^3}\right)dt+\left(\frac{g_1}{1+H_{t-}}-\frac{G_{t-}h_1}{(1+H_{t-})^2}\right)dW_t\\
&=\left(Y_{t-}^3G_{t-}h_1^2-Y_{t-}^2g_1h_1\right)dt+\left(Y_{t-}g_1-Y_{t-}^2G_{t-}h_1\right)dW_t\\
&=Y_{t-}(g_1(t)-Y_{t-}G_{t-}h_1(t))\left(dW_t-Y_{t-}h_1(t)\,dt\right).
\end{aligned}
\tag{59}
$$

For the terms related to the jumps of $d\left(\frac{G_t}{1+H_t}\right)$ in (58), we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}}\left(\frac{g_2}{1+H_{t-}}-\frac{G_{t-}h_2}{(1+H_{t-})^2}\right)(\mu-\nu)(dt,dx)\\
&+\int_{\mathbb{R}}\left(\frac{G_{t-}+g_2}{1+H_{t-}+h_2}-\frac{G_{t-}}{1+H_{t-}}-\left[\frac{g_2}{1+H_{t-}}-\frac{G_{t-}h_2}{(1+H_{t-})^2}\right]\right)(\mu-\nu)(dt,dx)\\
&+\int_{\mathbb{R}}\left(\frac{G_{t-}+g_2}{1+H_{t-}+h_2}-\frac{G_{t-}}{1+H_{t-}}-\left[\frac{g_2}{1+H_{t-}}-\frac{G_{t-}h_2}{(1+H_{t-})^2}\right]\right)\nu(dt,dx)\\
=&\int_{\mathbb{R}}\left(\frac{G_{t-}+g_2}{1+H_{t-}+h_2}-\frac{G_{t-}}{1+H_{t-}}\right)(\mu-\nu)(dt,dx)\\
&+\int_{\mathbb{R}}\left(\frac{G_{t-}+g_2}{1+H_{t-}+h_2}-\frac{G_{t-}}{1+H_{t-}}-\left[\frac{g_2}{1+H_{t-}}-\frac{G_{t-}h_2}{(1+H_{t-})^2}\right]\right)\nu(dt,dx).
\end{aligned}
$$

In view of Girsanov's theorem we can write this as

$$
\begin{aligned}
&\int_{\mathbb{R}}\left(\frac{G_{t-}+g_2}{1+H_{t-}+h_2}-\frac{G_{t-}}{1+H_{t-}}\right)(\mu-\widetilde{\nu})(dt,dx)\\
&+\int_{\mathbb{R}}\left(\frac{G_{t-}+g_2}{1+H_{t-}+h_2}-\frac{G_{t-}}{1+H_{t-}}\right)\widetilde{Y}(t,x)\nu(dt,dx)\\
&+\int_{\mathbb{R}}\left(\frac{G_{t-}h_2}{(1+H_{t-})^2}-\frac{g_2}{1+H_{t-}}\right)\nu(dt,dx),
\end{aligned}
\tag{60}
$$

where $\widetilde{\nu}(dt, dx)$ is a compensator of μ under some probability measure $\widetilde{\mathbb{P}} \overset{\text{loc}}{\ll} \mathbb{P}$. From the last equation it is evident that for $\frac{G_t}{1+H_t}$ to be a local martingale we must have

$$\int_{\mathbb{R}} \left(\frac{G_{t-} + g_2}{1 + H_{t-} + h_2} - \frac{G_{t-}}{1 + H_{t-}} \right) \widetilde{Y}(t,x)\nu(dt, dx) + \int_{\mathbb{R}} \left(\frac{G_{t-}h_2}{(1 + H_{t-})^2} - \frac{g_2}{1 + H_{t-}} \right) \nu(dt, dx) = 0. \tag{61}$$

From this we can determine the function $\widetilde{Y}(t, x)$ needed for the Girsanov change of measure:

$$\widetilde{Y}(t, x) = \frac{1 + H_{t-} + h_2}{1 + H_{t-}}. \tag{62}$$

The density process follows from Jacod and Shiryaev (1987, Theorem III.5.19). □

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References

Applebaum D. (2004). *Lévy Processes and Stochastic Calculus*. Cambridge University Press.

Bertoin J. (1996). *Lévy Processes*. Cambridge University Press.

Björk T., Di Masi G., Kabanov Y. and Runggaldier W. (1997). Towards a general theory of bond markets. *Finance and Stochastics* 1, 141-174.

Björk T., Kabanov Y. and Runggaldier W. (1997). Bond market structure in the presence of marked point processes. *Mathematical Finance* 7, 211-239.

Brigo, D. and Mercurio, F. (2001). *Interest Rate Models: Theory and Practice*. Springer, Berlin.

Carr, P. and Madan, D. B. (1999). Option valuation using the fast Fourier transform. *Journal of Computational Finance* 2(4), 61-73.

Eberlein, E. (2001). Application of generalized hyperbolic Lévy motions in finance. In: Barndorff-Nielsen, O.E, Mikosch, T. and Resnik, S.I. (Eds.), *Lévy Processes: Theory and Applications*, 319-336. Birkhäuser.

Eberlein, E. and Kluge, W. (2006). Exact pricing formulae for caps and swaptions in a Lévy term structure model. To appear in the *Journal of Computational Finance*.

Eberlein, E. and Koval, N. (2005). A cross-currency Lévy market model. Preprint, University of Freiburg.

Eberlein, E. and Raible, S., (1999). Term structure models driven by general Lévy processes. *Mathematical Finance* 9, 31-53.

Eberlein, E. and Özkan, F. (2005). The Lévy Libor model. *Finance and Stochastics* 9, 327-348.

Geman, H., El Karoui, N. and Rochet, J.C. (1995). Changes of numeraire, changes of probability measures and pricing of options. *Journal of Applied Probability* 32, 443-458.

Glasserman, P. and Kou, S.G. (2003). The term structure of simple forward rates with jump risk. *Mathematical Finance* 13, 383-410.

Heath, D., Jarrow, R. and Morton, A. (1992). Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica* 60, 77-105.

Hunt, P.J. and Kennedy, J.E. (2000). *Financial Derivatives in Theory and Practice*. Wiley Series in Probability and Statistics.

Jacod, J. and Shiryaev A.N. (1987). *Limit Theorems for Stochastic Processes*. Springer-Verlag.

Jamshidian, F. (1997). Libor and swap market models and measures. *Finance and Stochastics* 4, 293-330.

Jamshidian, F. (1999). Libor market model with semimartingales. Working paper, NetAnalytic Ltd.

Kallsen, J. and Shiryaev, A.N. (2002). The cumulant process and Esscher's change of measure. *Finance and Stochastics* 6, 397-428.

Lee, R. W. (2004). Option pricing by transform methods: extensions, unification, and error control. *Journal of Computational Finance* 7(3), 51-86.

Liinev, J. (2004). Topics in Mathematical Finance: from Interest Rate Models to Exotic Options. Ph.D. thesis, Ghent University.

Musiela, M. and Rutkowski, M. (1997a). Continuous–time term structure models: Forward measure approach. *Finance and Stochastics* 1, 261-292.

Musiela, M. and Rutkowski, M. (1997b). *Martingale Methods in Financial Modelling*. Springer-Verlag.

Pelsser, A. (2000). *Efficient Methods for Valuing Interest Rate Derivatives*. Springer-Verlag.

Raible, S. (2000). Lévy Processes in Finance: Theory, Numerics, and Empirical Facts. Ph.D. thesis, University of Freiburg.

Rutkowski, M. (1999). Models of forward Libor and swap rates. *Applied Mathematical Finance* 6, 29-60.

Rutkowski, M. (2001). Modelling of forward Libor and swap rates. Published in: Jouini, E., Cvitanić, J., and Musiela, M. (Eds.) Option Pricing, Interest Rates and Risk Management, Handbooks in Mathematical Finance, 336-395. Cambridge University Press.

Sato, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.

Schoutens, W. (2003). *Lévy Processes in Finance*. Wiley.