Valuation of floating range notes in Lévy term structure models*

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Abstract

Turnbull (1995) as well as Navatte and Quittard-Pinon (1999) derived explicit pricing formulae for digital options and range notes in a one-factor Gaussian Heath-Jarrow-Morton (henceforth HJM) model. Nunes (2004) extended their results to a multifactor Gaussian HJM framework. In this paper, we generalize these results by providing explicit pricing solutions for digital options and range notes in the multivariate Lévy term structure model of Eberlein and Raible (1999), that is an HJM-type model driven by a d-dimensional (possibly non-homogeneous) Lévy process. As a byproduct, we obtain a pricing formula for floating range notes in the special case of a multifactor Gaussian HJM model that is simpler than the one provided by Nunes (2004).

Key Words: Lévy process, term structure model, change of probability measure, bilateral Laplace transform, interest rate digital option, range note

1 Introduction

The main aim of this paper is to provide analytical valuation formulae for floating range notes in the Lévy term structure model introduced in Eberlein and Raible (1999) and pushed further in Eberlein and Özkan (2003), Eberlein, Jacod, and Raible (2005), and Eberlein and Kluge (2004). This model generalizes the multifactor Gaussian HJM model by replacing the driving Brownian motion with a multivariate (generally non-homogeneous) Lévy process.

Range notes are structured products, convenient for investors with a strong belief that interest rates will stay within a certain corridor. They provide interest payments which are proportional to the time in which a reference index rate (most commonly the Libor rate) lies inside that range. In return for the drawback that no interest will be paid for the time the corridor is left, they offer higher rates than comparable standard products, like e.g. floating rate notes. *Floating range notes* pay coupon rates which are linked to some reference index rate (e.g. 3-month Libor plus 100 basis points) whereas the coupon rates of *fixed range notes* are specified in advance. Let us stress that coupon payments of both products depend on the path of the reference index rate.

Turnbull (1995) provided an explicit valuation formula for floating range notes in the one-factor Gaussian HJM framework. Using the same model and the change-of-numeraire technique developed by Geman, El Karoui, and Rochet (1995), Navatte

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and Quittard-Pinon (1999) derived a pricing solution in a more intuitive way. For this purpose, they introduced *double delayed digital options*. The value of each floating range note coupon is shown to be equal to the value of a portfolio of those options plus some additional term. This extra term only involves the cumulative density function of a standard normal distribution. Nunes (2004) managed to generalize the former results to a multifactor Gaussian HJM model. His valuation formula for floating range notes looks very similar, i.e. each coupon is written as a portfolio of *delayed digital options* plus some extra term. This extra term, although given in closed form, is quite complicated and comes from evaluating the joint probability law of two random variables.

One purpose of this paper is to show that the calculation of the joint probability distribution can be circumvented by changing the probability measure in a suitable way. Proceeding this way, a much simpler pricing formula can be obtained in the multifactor Gaussian HJM model (see Theorem 5.4). However, our main goal is to price range notes in the more general framework of a Lévy term structure model. As a side result, a valuation formula for digital options is provided. Besides the change-of-numeraire technique we make use of integral transform methods. They are very useful tools whenever the characteristic function or bilateral Laplace transform of the underlying is known analytically. For option pricing these methods go back to Carr and Madan (1999) who use Fourier transforms and to Raible (2000) whose approach is based on bilateral Laplace transforms. In the context of deriving hedging strategies similar methods have been used by Hubalek and Krawczyk (1998).

The motivation for using a model driven by Lévy processes comes from a handicap of Gaussian models. In order to price exotic products, one should use a model that is consistent with market prices of plain vanilla options such as caps, floors, and swaptions. Gaussian models fail to reproduce the surface of implied volatilities of those options. Lévy models are more flexible and allow for a calibration to the market prices of caps and swaptions across different strikes and maturities with high accuracy. At the same time, they are still analytically tractable.

The outline of the paper is as follows. Section 2 gives a short introduction to the Lévy term structure model. Some tools that will be needed to price digital options and range notes are given in section 3. In particular, a new measure which is useful for pricing range notes is introduced. Section 4 is dedicated to the valuation of digital options. Explicit pricing solutions for range notes are provided in section 5.

2 Presentation of the Model

Let us briefly recall the HJM framework for modeling the term structure of interest rates. Subject to modeling are either zero coupon bond prices or instantaneous, continuously compounded forward rates. A *zero coupon bond* is a financial security that pays an amount of one currency unit to its owner at maturity T. We denote its price at time t by $P(t,T)$. Suppose that $T^* > 0$ is a fixed time horizon and assume that for every $T \in [0,T^*]$ there is a zero coupon bond maturing at T traded on the market. The *instantaneous forward rate* $f(t,T)$ is the forward rate at time t that applies for an infinitesimal time period starting at T. Formally, it is defined by $f(t,T) := -\frac{\partial}{\partial T}\log P(t,T)$. Zero coupon bond prices can be recovered from the forward rates via $P(t,T) = \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right)$. Thus, the term structure can be modeled by specifying either of them.

In the following, we give a short overview over the Lévy term structure model. For a detailed description including proofs we refer to Eberlein and Kluge (2004).

2.1 The driving process

The model is driven by a d-dimensional stochastic process $L = (L_t)_{0\leq t\leq T^*}$ with *independent increments and absolutely continuous characteristics*, abbreviated by *PIIAC*. These processes are also called *non-homogeneous Lévy processes*. More precisely, $L = (L^1,\ldots,L^d)$ is defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, has independent increments, and for every t the law of L_t is characterized by the characteristic function

$$\mathbb{E}\left[e^{i\langle u,L_t\rangle}\right] = \exp\int_0^t \left(i\langle u,b_s\rangle - \frac{1}{2}\langle u,c_s u\rangle + \int_{\mathbb{R}^d}(e^{i\langle u,x\rangle} - 1 - i\langle u,x\rangle)F_s(\mathrm{d}x)\right)\mathrm{d}s.$$

Here, $b_s \in \mathbb{R}^d$, c_s is a symmetric nonnegative-definite $d\times d$ matrix, and F_s is a measure on $\mathbb{R}^d$ that integrates $(|x|^2\wedge|x|)$ and satisfies $F_s(\{0\}) = 0$. The Euclidian scalar product on $\mathbb{R}^d$ is denoted by $\langle\cdot,\cdot\rangle$, the respective norm by $|\cdot|$. It is assumed that

$$\int_0^{T^*}\left(|b_s| + ||c_s|| + \int_{\mathbb{R}^d}(|x|^2\wedge|x|)F_s(\mathrm{d}x)\right)\mathrm{d}s < \infty$$

(where $||\cdot||$ denotes any norm on the set of $d\times d$ matrices) and that there are constants M, $\varepsilon > 0$ such that for every $u\in[-(1+\varepsilon)M,(1+\varepsilon)M]^d$

$$\int_0^{T^*}\int_{\{|x|>1\}}\exp\langle u,x\rangle F_s(\mathrm{d}x)\,\mathrm{d}s < \infty. \tag{2.1}$$

It is no restriction to require also that $\int_{\{|x|>1\}}\exp\langle u,x\rangle F_t(\mathrm{d}x) < \infty$ for all t. The latter assumption is equivalent to the existence of exponential moments of L, that is (2.1) holds if and only if $\mathbb{E}[\exp\langle u,L_t\rangle]$ is finite for $t\in[0,T^*]$ and $u\in[-(1+\varepsilon)M,(1+\varepsilon)M]^d$. We can conclude that L is an additive process in law and thus has a modification that is càdlàg, which means that all paths are right-continuous and admit left-hand limits (see e. g. Sato (1999, Theorem 11.5)). We will always work with this modification of L.

To simplify notation in what follows, let us denote by θ_s the cumulant associated with the infinitely divisible distribution characterized by the Lévy-Khintchine triplet (b_s,c_s,F_s), i.e. for $z\in[-(1+\varepsilon)M,(1+\varepsilon)M]^d$

$$\theta_s(z) := \langle z,b_s\rangle + \frac{1}{2}\langle z,c_s z\rangle + \int_{\mathbb{R}^d}(e^{\langle z,x\rangle} - 1 - \langle z,x\rangle)F_s(\mathrm{d}x).$$

We can extend θ_s to complex numbers $z\in\mathbb{C}^d$ with $\Re(z_j)\in[-(1+\varepsilon)M,(1+\varepsilon)M]$ for $j\in\{1,\ldots,d\}$ and write the characteristic function of L_t as

$$\mathbb{E}\left[e^{i\langle u,L_t\rangle}\right] = \exp\int_0^t\theta_s(iu)\,\mathrm{d}s. \tag{2.2}$$

Note that $iu := (iu_j)_{1\leq j\leq d}$ and the scalar product on $\mathbb{R}^d$ is extended to complex numbers, that is $\langle w,z\rangle = \sum_{j=1}^d w_j z_j$ for $w,z\in\mathbb{C}^d$. However, $\langle\cdot,\cdot\rangle$ is not the Hermitian scalar product. If L is a (homogeneous) Lévy process, i.e. the increments of L are stationary, b_s, c_s, and F_s and thus also θ_s do not depend on s. In this case we will write θ for short. θ then equals the cumulant (also called log moment generating function) of L_1.

We endow the probability space with a filtration $(\mathcal{F}_s)_{0\leq s\leq T^*}$: Let $\mathcal{F} = \mathcal{F}_{T^*}$ and suppose that $(\mathcal{F}_s)_{0\leq s\leq T^*}$ is the smallest right continuous filtration to which L is adapted. Then L is a special semimartingale with respect to this filtration and its characteristics in the sense of Jacod and Shiryaev (2003, Chapter II, Definition 2.6) are given by

$$A_t = \int_0^t b_s \,\mathrm{d}s, \quad C_t = \int_0^t c_s \,\mathrm{d}s, \quad \nu(\mathrm{d}s, \mathrm{d}x) = F_s(\mathrm{d}x)\,\mathrm{d}s.$$

These characteristics allow us to write L in its so-called canonical representation (see Jacod and Shiryaev (2003, II.2.38))

$$L_t = \int_0^t b_s \,\mathrm{d}s + L_t^c + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu)(\mathrm{d}s, \mathrm{d}x).$$

Here, L^c denotes the continuous martingale part of L and μ is the random measure associated with the jumps of L. From the characteristic C we can conclude that $L_t^c = \int_0^t \sqrt{c_s}\,\mathrm{d}W_s$, where W is a standard d-dimensional Brownian motion and $\sqrt{c_s}$ is a measurable version of the square root of c_s.

2.2 The dynamics of the forward rates

The dynamics of the instantaneous forward rates for $T \in [0, T^*]$ are given by

$$f(t,T) = f(0,T) + \int_0^t \partial_2 A(s,T)\,\mathrm{d}s - \int_0^t \partial_2 \Sigma(s,T)\,\mathrm{d}L_s \qquad (0 \leq t \leq T),$$

where ∂_2 denotes the derivation operator with respect to the second argument. The initial values $f(0,T)$ are deterministic, measurable, and bounded in T. Σ and A are deterministic functions with values in $\mathbb{R}^d$ and $\mathbb{R}$ respectively defined on $\Delta := \{(s,T) \in [0,T^*] \times [0,T^*] : s \leq T\}$ whose paths are continuously differentiable in the second variable. Moreover, they satisfy the following conditions:

1. The volatility structure Σ is bounded in the following way:
   for $(s,T) \in \Delta$ we have
   $$0 \leq \Sigma^i(s,T) \leq M \qquad (i \in \{1, \ldots, d\})$$
   where M is the constant from (2.1).
   Furthermore, $\Sigma(s,T) \neq 0$ for $s < T$ and $\Sigma(T,T) = 0$ for $T \in [0,T^*]$.

2. The drift coefficients $A(\cdot, T)$ are given by
   $$A(s,T) = \theta_s(\Sigma(s,T)). \tag{2.3}$$

Remark: Drift condition (2.3) guarantees that bond prices, once discounted by the money market account, are martingales. Thus, the model works directly under a martingale measure. If the dimension of the driving process L is $d = 1$, the martingale measure $\mathbb{P}$ is unique. For a discussion of the uniqueness of the martingale measure we refer to Eberlein, Jacod, and Raible (2005). If there is more than one martingale measure the problem of which one to choose arises. In this case, we assume that $\mathbb{P}$ is the risk-neutral measure chosen by the market and price integrable contingent claims by taking the $\mathbb{P}$-expectation of the discounted payoffs.

From the forward rates we can deduce explicit expressions for zero coupon bond prices and the money market account $B_t := \exp \int_0^t r(s)\,\mathrm{d}s$, where $r(s) := f(s,s)$ denotes the short rate:

$$P(t,T) = P(0,T) \exp\left(\int_0^t (r(s) - A(s,T))\,\mathrm{d}s + \int_0^t \Sigma(s,T)\,\mathrm{d}L_s \right). \tag{2.4}$$

Setting $T = t$ and using $P(t,t) = 1$ yields

$$B_t = \frac{1}{P(0,t)} \exp\left(\int_0^t A(s,t)\,\mathrm{d}s - \int_0^t \Sigma(s,t)\,\mathrm{d}L_s \right) \tag{2.5}$$

and we get another representation of the bond price that will be useful later:

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left(- \int_0^t A(s,t,T)\,\mathrm{d}s + \int_0^t \Sigma(s,t,T)\,\mathrm{d}L_s \right), \tag{2.6}$$

where we used the abbreviations

$$A(s,t,T) := A(s,T) - A(s,t)$$

and

$$\Sigma(s,t,T) := \Sigma(s,T) - \Sigma(s,t). \tag{2.7}$$

3 Tools for option valuation

To price digital options and range notes we will mainly use two techniques. First, we will change the numeraire and switch from the spot martingale measure $\mathbb{P}$ to a forward martingale measure or some measure that we will call adjusted forward measure. Second, the option price will be expressed as a convolution. We then perform a Laplace transformation followed by an inverse Laplace transformation. This procedure is useful because the Laplace transform of a convolution equals the product of the Laplace transforms of the convolution factors, which are easy to calculate.

Remember that the *forward martingale measure* for the settlement day T, denoted by $\mathbb{P}_T$, is defined by the Radon-Nikodym derivative

$$\frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\mathbb{P}} := \frac{1}{B_T P(0,T)}.$$

Usually, this measure is defined on $(\Omega, \mathcal{F}_T)$ only, but we can and do define it on $(\Omega, \mathcal{F}_{T^*})$. $\mathbb{P}$ and $\mathbb{P}_T$ are equivalent and from (2.5) we get the explicit expression

$$\frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\mathbb{P}} = \exp\left(-\int_0^T A(s,T)\,\mathrm{d}s + \int_0^T \Sigma(s,T)\,\mathrm{d}L_s\right). \tag{3.1}$$

Restricted to the σ-field $\mathcal{F}_t$ for $t \leq T$ this becomes

$$\begin{aligned}\frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t} &= \mathbb{E}_{\mathbb{P}}\left[\frac{1}{B_T P(0,T)}\bigg|\mathcal{F}_t\right] = \frac{P(t,T)}{B_t P(0,T)} \\ &= \exp\left(-\int_0^t A(s,T)\,\mathrm{d}s + \int_0^t \Sigma(s,T)\,\mathrm{d}L_s\right). \end{aligned}\tag{3.2}$$

The two predictable processes in Girsanov's theorem for semimartingales (see Jacod and Shiryaev (2003, Theorem III.3.24)) associated with this change of measure are (compare Eberlein and Kluge (2004, Proposition 10))

$$\begin{aligned}\beta(s) &= \Sigma(s,T) \qquad \text{and} \\ Y(s,x) &= \exp\langle \Sigma(s,T), x\rangle.\end{aligned}$$

With their help, the semimartingale characteristics of L under $\mathbb{P}_T$ can be obtained. In particular, L remains a PIIAC and a special semimartingale.

For $T' < T$ we define the *adjusted forward measure* $\mathbb{P}_{T',T}$ on $(\Omega, \mathcal{F}_{T^*})$ via

$$\frac{\mathrm{d}\mathbb{P}_{T',T}}{\mathrm{d}\mathbb{P}_T} := \frac{F(T',T',T)}{F(0,T',T)} = \frac{P(0,T)}{P(0,T')P(T',T)}, \tag{3.3}$$

where $F(\cdot,T',T) := \frac{P(\cdot,T')}{P(\cdot,T)}$ denotes the forward price process. Restricting this density to $\mathcal{F}_t$ for $t \leq T'$ we get

$$\frac{\mathrm{d}\mathbb{P}_{T',T}}{\mathrm{d}\mathbb{P}_T}\bigg|_{\mathcal{F}_t} = \frac{F(t,T',T)}{F(0,T',T)} = \frac{P(0,T)P(t,T')}{P(0,T')P(t,T)} \tag{3.4}$$

since $(F(t,T',T))_{0\leq t\leq T'}$ is a $\mathbb{P}_T$-martingale. Thus we have

$$\frac{\mathrm{d}\mathbb{P}_{T',T}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t} = \frac{P(0,T)P(t,T')}{P(0,T')P(t,T)}\,\frac{P(t,T)}{B_t P(0,T)} = \frac{P(t,T')}{B_t P(0,T')},$$

i.e. the forward measure $\mathbb{P}_{T'}$ and the adjusted forward measure $\mathbb{P}_{T',T}$ are equal once restricted to $(\Omega, \mathcal{F}_t)$ for $t \leq T'$. However, on $(\Omega, \mathcal{F}_t)$ for $t > T'$ they are usually different. Choose for example $T' < t < T$, then in general

$$\begin{aligned}\frac{\mathrm{d}\mathbb{P}_{T',T}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t} &= \frac{P(t,T)}{P(T',T)B_t P(0,T')} \\ &\overset{(2.4)}{=} \frac{1}{B_{T'}P(0,T')}\exp\left(-\int_{T'}^t A(s,T)\,\mathrm{d}s + \int_{T'}^t \Sigma(s,T)\,\mathrm{d}L_s\right) \\ &\neq \frac{1}{B_{T'}P(0,T')} \\ &= \frac{\mathrm{d}\mathbb{P}_{T'}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_t}.\end{aligned}$$

The adjusted forward measure will prove to be very useful to price range notes. Also, the following proposition will be needed.

Proposition 3.1 *Suppose that $f : \mathbb{R}_+ \to \mathbb{C}^d$ is a continuous function such that $|\Re(f^i(x))| \leq M$ for all $i \in \{1, \dots, d\}$ and $x \in \mathbb{R}_+$, then*

$$\mathbb{E}\left[\exp\left(\int_t^T f(s)\,\mathrm{d}L_s\right)\right] = \exp\int_t^T \theta_s(f(s))\,\mathrm{d}s.$$

(The integrals are to be understood componentwise for real and imaginary part.)

Proof: This proposition can be proved in the same way as in Eberlein and Raible (1999, Lemma 3.1). □

Remember that in case L is a homogeneous Lévy process, θ_s does not depend on s and equals the log moment generating function of L_1.

4 Digital Options

In this section, we discuss the pricing of interest rate digital options. For convenience we adopt the notation of Nunes (2004).

A *standard European interest rate digital call (put)* with strike rate r_k is a financial security that pays an amount of one currency unit to its owner if and only if the simply compounded interest rate for the period $[T, T+\delta]$ lies above (below) r_k at maturity T of the option. More precisely, the time-T price of this option is given by

$$SD(\Theta)_T[r_n(T, T+\delta); r_k; T] := \mathbb{1}_{\{\Theta r_n(T,T+\delta) > \Theta r_k\}},$$

with

$$r_n(T, T+\delta) := \frac{1}{\delta}\left[\frac{1}{P(T, T+\delta)} - 1\right], \tag{4.1}$$

and where $\Theta = 1$ for a digital call and $\Theta = -1$ for a digital put.

In accordance with Navatte and Quittard-Pinon (1999) and Nunes (2004) we call an interest rate digital option *delayed* if option maturity T and payment date T_1 differ ($T_1 > T$). The time-T_1 price of a *delayed digital option* is given by

$$DD(\Theta)_{T_1}[r_n(T, T+\delta); r_k; T_1] := \mathbb{1}_{\{\Theta r_n(T,T+\delta) > \Theta r_k\}},$$

where again $\Theta = 1$ for a delayed digital call and $\Theta = -1$ for a delayed digital put. Since a standard digital option is a special case of a delayed digital option ($T_1 = T$), we will only consider the latter in the following.

Delayed range digital options provide a terminal payoff equal to 1 paid at T_1 if and only if at option maturity T ($T \leq T_1$) the underlying interest rate lies inside a prespecified corridor. Consequently, the time-T_1 price of a delayed range digital option is

$$DRD_{T_1}[r_n(T, T+\delta); r_l; r_u; T_1] := \mathbb{1}_{\{r_n(T,T+\delta) \in [r_l, r_u]\}}.$$

By arbitrage arguments, the time-t prices ($t \in [0, T_1]$) of delayed digital calls, puts, and range options are related via

$$\begin{aligned} DRD_t[r_n(T, T+\delta); r_l; r_u; T_1] &= P(t, T_1) - DD(1)_t[r_n(T, T+\delta); r_u; T_1] \\ &\quad - DD(-1)_t[r_n(T, T+\delta); r_l; T_1]. \end{aligned}$$

Unfortunately, a call-put parity like

$$DD(1)_t[r_n(T,T+\delta);r_k;T_1] = P(t,T_1) - DD(-1)_t[r_n(T,T+\delta);r_k;T_1] \tag{4.2}$$

does not hold for all $t \in [0,T_1]$ (note that in case $r_n(T,T+\delta) = r_k$ equality fails for $t \in [T,T_1]$). However, equation (4.2) holds for $t < T$ in models where the distribution of $P(T,T+\delta)$ does not have point masses (like e.g. in the Gaussian HJM model with a reasonable volatility structure). If L is a Poisson process, equation (4.2) might fail for $t < T$ though. The technique that we are going to present for option valuation only works for model specifications that do not produce point masses in the distribution of $P(T,T+\delta)$ (see Proposition 4.1 and the discussion preceding it). In these cases, we have the call-put parity (4.2) for $t < T$ and can thus price any of the mentioned digital options if we are able to price a delayed digital call.

We calculate the value of the call by taking the $\mathbb{P}$-conditional expectation of its discounted payoff, i.e.

$$\begin{aligned}
D_t &:= DD(1)_t[r_n(T,T+\delta);r_k;T_1] \\
&= B_t \mathbb{E}\left[\frac{1}{B_{T_1}} 1\!\!1_{\{r_n(T,T+\delta)>r_k\}} \Big| \mathcal{F}_t\right] \\
&= P(t,T_1)\mathbb{E}_{T_1}\left[1\!\!1_{\{r_n(T,T+\delta)>r_k\}} \Big| \mathcal{F}_t\right] \\
&\overset{(4.1)}{=} P(t,T_1)\mathbb{E}_{T_1}\left[1\!\!1_{\left\{P(T,T+\delta)<\frac{1}{\delta r_k+1}\right\}} \Big| \mathcal{F}_t\right] \\
&\underset{(2.6)}{\overset{(2.4)}{=}} P(t,T_1)\mathbb{E}_{T_1}\left[1\!\!1_{\left\{\frac{P(t,T+\delta)}{P(t,T)}\exp\left[-\int_t^T A(s,T,T+\delta)\,\mathrm{d}s + \int_t^T \Sigma(s,T,T+\delta)\,\mathrm{d}L_s\right]<\frac{1}{\delta r_k+1}\right\}} \Big| \mathcal{F}_t\right].
\end{aligned}$$

$\mathbb{E}_{T_1}$ denotes the expectation with respect to the forward measure $\mathbb{P}_{T_1}$. For the change of numeraire we used equations (3.1)-(3.2) and the abstract Bayes formula. By independence of the increments of L and since $\frac{P(t,T+\delta)}{P(t,T)}$ is $\mathcal{F}_t$-measurable, we get (compare Musiela and Rutkowski (1998, Lemma A.0.1.(v)))

$$D_t = P(t,T_1)h\left(\frac{P(t,T+\delta)}{P(t,T)}\right) \tag{4.3}$$

with $h : \mathbb{R} \to [0,1]$ given by

$$h(y) := \mathbb{E}_{T_1}\left[1\!\!1_{\left\{y\exp\left[-\int_t^T A(s,T,T+\delta)\,\mathrm{d}s + \int_t^T \Sigma(s,T,T+\delta)\,\mathrm{d}L_s\right]<\frac{1}{\delta r_k+1}\right\}}\right].$$

To calculate $h(y)$ for $y > 0$, observe that

$$h(y) = \int 1\!\!1_{\left\{e^X<\frac{K}{y}\right\}}\,\mathrm{d}\mathbb{P}_{T_1} = \int 1\!\!1_{\left\{e^x<\frac{K}{y}\right\}}\,\mathrm{d}\mathbb{P}_{T_1}^X(x) \tag{4.4}$$

where

$$\begin{aligned}
X &:= \int_t^T \Sigma(s,T,T+\delta)\,\mathrm{d}L_s, \\
K &:= \frac{1}{\delta r_k+1}\exp\int_t^T A(s,T,T+\delta)\,\mathrm{d}s,
\end{aligned} \tag{4.5}$$

and $\mathbb{P}_{T_1}^X$ denotes the distribution of X under $\mathbb{P}_{T_1}$. If this distribution possesses a Lebesgue-density φ in $\mathbb{R}$ then

$$
\begin{aligned}
h(y) &= \int \mathbb{1}_{\left\{e^x < \frac{K}{y}\right\}} \varphi(x)\,\mathrm{d}x \\
&= \int f_y(-x)\varphi(x)\,\mathrm{d}x \\
&= (f_y * \varphi)(0) = V(0) \qquad (4.6)
\end{aligned}
$$

with $f_y(x) := \mathbb{1}_{\left\{e^{-x} < \frac{K}{y}\right\}}(x)$ and $V(\xi) := (f_y * \varphi)(\xi)$.

Before deriving a formula for the option price, let us shortly discuss the assumption that $\mathbb{P}_{T_1}^X$ possesses a Lebesgue-density. This distribution has a Lebesgue-density if and only if it is absolutely continuous (with respect to the Lebesgue-measure on $\mathbb{R}$). Since $\mathbb{P}$ and $\mathbb{P}_{T_1}$ are equivalent, this is the case if and only if the distribution of X with respect to $\mathbb{P}$, denoted by $\mathbb{P}^X$, is absolutely continuous. Whether or not $\mathbb{P}^X$ is absolutely continuous cannot be answered in general. The answer depends on the choice of the volatility structure and the driving process, as the following examples show:

1. Let $\Sigma(s, T+\delta) = \Sigma(s, T)$ for $s \in [t, T]$ (i.e. $\Sigma(s, T, T+\delta) = 0$), then $X = 0$ and $\mathbb{P}^X$ cannot be absolutely continuous.

2. Choose the Ho-Lee volatility structure, i.e. $\Sigma(s, T) = \hat{\sigma}(T - s)$, and let L be a Poisson process, then $X = \hat{\sigma}\delta(L_T - L_t)$, whose distribution is not absolutely continuous since the distribution of L_{T-t} is not.

The following proposition gives sufficient conditions for absolute continuity.

Proposition 4.1 *Assume that $\Sigma(s, T, T+\delta) \neq 0$ for $s \in [t, T]$. Then each of the following conditions implies that $\mathbb{P}^X$ is absolutely continuous with respect to the Lebesgue-measure λ:*

1. *There is a Borel set $S \subset [t, T]$ with $\lambda(S) > 0$ such that c_s is positive definite for $s \in S$.*

2. *Denote by Φ_s the characteristic function associated with the Lévy-Khintchine triplet (b_s, c_s, F_s), i.e. for $u \in \mathbb{R}^d$*

$$
\begin{aligned}
\Phi_s(u) &= \exp\left(i\langle u, b_s\rangle - \frac{1}{2}\langle u, c_s u\rangle + \int_{\mathbb{R}^d} (e^{i\langle u, x\rangle} - 1 - i\langle u, x\rangle) F_s(\mathrm{d}x)\right) \\
&= \exp(\theta_s(iu)).
\end{aligned}
$$

Then

$$|\Phi_s(u)| \leq C \exp\left(-\gamma |u|^\eta\right) \qquad (t \leq s \leq T)$$

for real constants C, γ, $\eta > 0$ that do not depend on s.

Proof: We show that Φ^X, i.e. the characteristic function of X, is integrable. Using Proposition 3.1 we get

$$\Phi^X(u) = \exp \int_t^T \theta_s(iu\Sigma(s, T, T+\delta))\,\mathrm{d}s \qquad (u \in \mathbb{R}). \qquad (4.7)$$

Let us first suppose that condition 1 is satisfied and define L^1 and L^2 by

$$\begin{aligned} L_t^1 &:= \int_0^t b_s\,\mathrm{d}s + \int_0^t \sqrt{c_s}\,\mathrm{d}W_s, \\ L_t^2 &:= \int_0^t \int_{\mathbb{R}^d} x(\mu-\nu)(\mathrm{d}s,\mathrm{d}x). \end{aligned}$$

Both processes are PIIACs and $L = L^1 + L^2$. By (4.7) and using Proposition 3.1 on L^1 and L^2 we have

$$\Phi^X(u) = \Phi^{X^1}(u)\Phi^{X^2}(u),$$

where $\Phi^{X^j}(u) := \mathbb{E}\left[\exp\left(iu\int_t^T \Sigma(s,T,T+\delta)\,\mathrm{d}L_s^j\right)\right]$ for $j \in \{1,2\}$. Since both factors are bounded above by 1, it is enough to show that one of them is integrable. But

$$\begin{aligned} \left|\Phi^{X^1}(u)\right| &= \exp\left(-\frac{1}{2}u^2\int_t^T \langle \Sigma(s,T,T+\delta), c_s\Sigma(s,T,T+\delta)\rangle\,\mathrm{d}s\right) \\ &=: \exp\left(-\frac{1}{2}u^2 C\right) \end{aligned}$$

where $C > 0$ due to the positive definiteness of c_s for $s \in S$ and the fact that $\lambda(S) > 0$. Hence Φ^{X^1} is integrable.

Now suppose condition 2 is satisfied. Then by (4.7)

$$\begin{aligned} |\Phi^X(u)| &= \exp\int_t^T \Re\Big(\theta_s(iu\Sigma(s,T,T+\delta))\Big)\,\mathrm{d}s \\ &= \exp\int_t^T \log|\Phi_s(u\Sigma(s,T,T+\delta))|\,\mathrm{d}s \\ &\le \exp\int_t^T \log\Big(C\exp(-\gamma|u\Sigma(s,T,T+\delta)|^\eta)\Big)\,\mathrm{d}s \\ &= C^{T-t}\exp\left(-\gamma|u|^\eta\int_t^T |\Sigma(s,T,T+\delta)|^\eta\,\mathrm{d}s\right) \\ &=: C^{T-t}\exp\left(-\widetilde{\gamma}|u|^\eta\right), \end{aligned}$$

where $\widetilde{\gamma} > 0$ since $\Sigma(s,T,T+\delta) \neq 0$ for $s \in [t,T]$. Consequently, Φ^X is integrable. $\Box$

Let us come back to option pricing and denote by $M_{T_1}^X$ the moment generating function of the random variable X with respect to the measure $\mathbb{P}_{T_1}$. The next theorem gives an analytic expression for the price of the call that can be evaluated numerically very fast.

Theorem 4.2 *Suppose the distribution of X possesses a Lebesgue-density. Choose an $R > 0$ such that $M_{T_1}^X(-R) < \infty$. Then*

$$D_t = \frac{1}{\pi}P(t,T_1)\int_0^\infty \Re\left(\left(\frac{P(t,T)}{P(t,T+\delta)}K\right)^{R+iu}\frac{1}{R+iu}M_{T_1}^X(-R-iu)\right)\mathrm{d}u$$

with

$$K := \frac{1}{\delta r_k + 1}\exp\int_t^T A(s,T,T+\delta)\,\mathrm{d}s.$$

REMARK: It is always possible to choose an R that satisfies the prerequisites of the theorem (compare Eberlein and Kluge (2004, Lemma 13)). The particular choice of R does of course not have an impact on the option price, but it has influence on the speed at which the integral can be evaluated numerically (see Raible (2000, Section 3.7)).

PROOF: We use the convolution representation (4.6) and apply Raible (2000, Theorem B.2) to the functions $F_1(x) := f_y(x)$ and $F_2(x) := \varphi(x)$, that is we express the bilateral Laplace transform of their convolution as the product of the bilateral Laplace transforms of the factors. The prerequisites of the theorem are satisfied since $x \mapsto e^{-Rx} f_y(x)$ is bounded,

$$\int_{\mathbb{R}} e^{-Rx} |f_y(x)| \,\mathrm{d}x = \frac{1}{R} \left(\frac{K}{y} \right)^R < \infty,$$

and

$$\int_{\mathbb{R}} e^{-Rx} |\varphi(x)| \,\mathrm{d}x = M_{T_1}^X(-R) < \infty.$$

The cited theorem together with (4.6) yields

$$L[V](R+iu) = L[f_y](R+iu) L[\varphi](R+iu) \qquad (u \in \mathbb{R})$$

where $L[\cdot]$ denotes the bilateral Laplace transform. The theorem also yields that $\xi \mapsto V(\xi)$ is continuous and that $\int_{\mathbb{R}} e^{-R\xi} V(\xi) \,\mathrm{d}\xi$ is absolutely convergent. Therefore, we may apply Raible (2000, Theorem B.3) and get

$$\begin{aligned} V(0) &= \frac{1}{2\pi i} \lim_{Y \to \infty} \int_{R-iY}^{R+iY} L[V](z) \,\mathrm{d}z \\ &= \frac{1}{2\pi} \lim_{Y \to \infty} \int_{-Y}^{Y} L[V](R+iu) \,\mathrm{d}u \\ &= \frac{1}{2\pi} \lim_{Y \to \infty} \int_{-Y}^{Y} L[f_y](R+iu) L[\varphi](R+iu) \,\mathrm{d}u, \end{aligned}$$

if this limit exists. Note that the integrand evaluated at u equals the complex conjugate of the integrand evaluated at $-u$. Therefore, using the relationship $z+\bar{z} = 2\,\Re(z)$ one arrives at

$$V(0) = \frac{1}{\pi} \lim_{Y \to \infty} \int_0^Y \Re\Big(L[f_y](R+iu) L[\varphi](R+iu) \Big) \,\mathrm{d}u.$$

We have

$$L[\varphi](R+iu) = M_{T_1}^X(-R-iu)$$

and, since $R > 0$, one obtains

$$L[f_y](R+iu) = \frac{1}{(R+iu)} \left(\frac{K}{y} \right)^{R+iu}$$

and (after some calculations) concludes that the above limit exists. Plugging in the expressions from (4.3),(4.5), and (4.6) yields the claim. □

Theorem 4.3 *Under the assumptions of Theorem 4.2 we have an explicit expression for* $M_{T_1}^X$*, namely for* $u \in \mathbb{R}$

$$M_{T_1}^X(-R-iu) = \exp \int_t^T \Big[\theta_s(g_s(-R-iu)) - \theta_s(g_s(0))\Big] \,\mathrm{d}s \tag{4.8}$$

with $g_s(z) := z\Sigma(s,T,T+\delta) + \Sigma(s,T_1)$.

Proof: To obtain the expression for the moment generating function of X we use equation (3.1), the independence of the increments of L, the fact that we have $\mathbb{E}[\exp \int_t^T \Sigma(s,T)\,\mathrm{d}L_s] = \exp \int_t^T A(s,T)\,\mathrm{d}s$, equation (2.3), and Proposition 3.1 (in this order) and get for $z \in \mathbb{C}$ with $\Re(z) = -R$

$$\begin{aligned}
M_{T_1}^X(z) &= \mathbb{E}_{T_1}\left[\exp\left(z\int_t^T \Sigma(s,T,T+\delta)\,\mathrm{d}L_s\right)\right] \\
&= \exp\left(-\int_0^{T_1} A(s,T_1)\,\mathrm{d}s\right) \\
&\quad \times\mathbb{E}\left[\exp\left(z\int_t^T \Sigma(s,T,T+\delta)\,\mathrm{d}L_s + \int_0^{T_1}\Sigma(s,T_1)\,\mathrm{d}L_s\right)\right] \\
&= \exp\left(-\int_t^T A(s,T_1)\,\mathrm{d}s\right) \\
&\quad \times\mathbb{E}\left[\exp\left(\int_t^T \Big(z\Sigma(s,T,T+\delta)+\Sigma(s,T_1)\Big)\,\mathrm{d}L_s\right)\right] \\
&= \exp\left(-\int_t^T \theta_s(g_s(0))\,\mathrm{d}s\right)\exp\left(\int_t^T \theta_s(g_s(z))\,\mathrm{d}s\right)
\end{aligned}$$

with $g_s(z) := z\Sigma(s,T,T+\delta) + \Sigma(s,T_1)$. Now (4.8) follows. □

Let us consider the multifactor Gaussian HJM model as a special case, i.e. L is a d-dimensional standard Brownian motion under $\mathbb{P}$. Then $\theta(x) = \frac{\langle x,x\rangle}{2}$ for $x \in \mathbb{C}^d$. From (4.8) and using (2.7) we get for $z \in \mathbb{C}$

$$\begin{aligned}
M_{T_1}^X(z) &= \exp\left(\frac{z^2}{2}\int_t^T ||\Sigma(s,T+\delta)-\Sigma(s,T)||^2\,\mathrm{d}s\right. \\
&\quad \left. +z\int_t^T \langle \Sigma(s,T+\delta)-\Sigma(s,T),\Sigma(s,T_1)\rangle\,\mathrm{d}s\right).
\end{aligned}$$

Consequently, X is normally distributed under $\mathbb{P}_{T_1}$ with mean

$$m(t,T,T+\delta,T_1) := \int_t^T \langle \Sigma(s,T+\delta)-\Sigma(s,T),\Sigma(s,T_1)\rangle\,\mathrm{d}s$$

and variance

$$g(t,T,T+\delta) := \int_t^T ||\Sigma(s,T+\delta)-\Sigma(s,T)||^2\,\mathrm{d}s.$$

From (4.4) we get

$$h(y) = \int_{-\infty}^{\log\frac{K}{y}} \mathrm{d}\mathbb{P}_{T_1}^X(x) = \mathbb{P}_{T_1}\left(X \le \log\frac{K}{y}\right) = \Phi\left(\frac{\log\frac{K}{y} - m(t,T,T+\delta,T_1)}{\sqrt{g(t,T,T+\delta)}}\right),$$

where Φ denotes the cumulative density function of a standard normal distribution. Plugging in the expression for K from (4.5) and using (4.3) we end up with

$$D_t = P(t, T_1)\Phi\left(\frac{\log\frac{P(t,T)}{P(t,T+\delta)(\delta r_k+1)} + \frac{1}{2}g(t, T, T+\delta) - l(t, T, T+\delta, T_1)}{\sqrt{g(t, T, T+\delta)}}\right),$$

where

$$l(t, T, T+\delta, T_1) := \int_t^T \langle \Sigma(s, T+\delta) - \Sigma(s, T), \Sigma(s, T_1) - \Sigma(s, T)\rangle \,\mathrm{d}s.$$

This formula coincides with the one derived in Nunes (2004, Proposition 3.3). Note that for a standard digital call ($T_1 = T$) one gets $l(t, T, T+\delta, T_1) = 0$.

5 Range Notes

The purpose of this section is to derive a formula for pricing range notes in the Lévy term structure model. As a special case, we will also consider the multifactor Gaussian HJM model and obtain a pricing formula that is simpler than the one provided by Nunes (2004). Once again, his notation is adopted.

In the following, we put ourselves at time t, the valuation date of the range note. Consider a bond with bullet redemption having had its previous coupon payment date at T_0 ($\leq t$) and having its N future coupons paid at times T_{j+1} ($j = 0, \ldots, N-1$). Based on some day count convention, let n_j (δ_j) denote the number of days (years) between the times T_j and T_{j+1}. For the current period we split up n_0 into the sum of n_0^- and n_0^+, representing the number of days between T_0 and t and between t and T_1 respectively. Furthermore, denote by $T_{j,i}$ the date that corresponds to i days after T_j and by $\delta_{j,i}$ the length (in years) of the compounding period starting at time $T_{j,i}$.

5.1 Fixed Range Notes

For the multifactor Gaussian HJM model, Nunes (2004) shows that the value of a fixed range note equals the value of a portfolio of delayed range digital options. Although the Lévy term structure model is more general, the same arguments apply since they do not depend on the driving process. We refer the reader to Nunes (2004, Proposition 4.1).

5.2 Floating Range Notes

To value floating range notes we will first switch from the spot measure to a suitable forward measure. Afterwards, another change of measure from the forward measure to an adjusted forward measure will be performed. Proceeding this way, we will not have to deal with a joint probability distribution of two random variables.

We cite the following definition from Nunes (2004):

Definition 5.1 *For a floating range note, the value of the $(j+1)^{th}$ coupon, at time T_{j+1}, is equal to*

$$\nu_{j+1}(T_{j+1}) := \frac{r_n(T_j, T_j + \delta_j) + s_j}{D_j} H(T_j, T_{j+1}),$$

where s_j represents the spread over the reference interest rate paid by the bond during the $(j+1)^{th}$ compounding period, D_j is the number of days in a year for the $(j+1)^{th}$ compounding period, and

$$H(T_j, T_{j+1}) := \sum_{i=1}^{n_j} \mathbb{1}_{\{r_l(T_{j,i}) \le r_n(T_{j,i}, T_{j,i}+\delta_{j,i}) \le r_u(T_{j,i})\}}$$

denotes the number of days, in the $(j+1)^{th}$ compounding period, that the reference interest rate lies inside a prespecified range, which is equal to $[r_l(T_{j,i}), r_u(T_{j,i})]$ for the i^{th} day of the $(j+1)^{th}$ compounding period.

Consequently, the time-t value of the floating range note is given by

$$FlRN(t) := P(t, T_N) + \sum_{j=0}^{N-1} \nu_{j+1}(t)$$

where $P(t, T_N)$ corresponds to the discounted value of the final payment of 1.

For the valuation of the first coupon we follow Nunes (2004) and get, since $r_n(T_0, T_0 + \delta_0)$ is already known at time t or, mathematically speaking, measurable with respect to $\mathcal{F}_t$,

$$\begin{aligned}
\nu_1(t) &= B_t \mathbb{E}\left[\frac{1}{B_{T_1}} \frac{r_n(T_0, T_0+\delta_0) + s_0}{D_0} H(T_0, T_1) \middle| \mathcal{F}_t\right] \\
&= \frac{r_n(T_0, T_0+\delta_0) + s_0}{D_0} P(t, T_1) \mathbb{E}_{T_1}\left[H(T_0, T_1) \middle| \mathcal{F}_t\right] \\
&= \frac{r_n(T_0, T_0+\delta_0) + s_0}{D_0} \Bigg(P(t, T_1) H(T_0, t) \\
&\quad + \sum_{i=n_0^-+1}^{n_0} P(t, T_1) \mathbb{E}_{T_1}\left[\mathbb{1}_{\{r_l(T_{0,i}) \le r_n(T_{0,i}, T_{0,i}+\delta_{0,i}) \le r_u(T_{0,i})\}} \middle| \mathcal{F}_t\right] \Bigg) \\
&= \frac{r_n(T_0, T_0+\delta_0) + s_0}{D_0} \Bigg(P(t, T_1) H(T_0, t) \\
&\quad + \sum_{i=n_0^-+1}^{n_0} DRD_t\left[r_n(T_{0,i}, T_{0,i}+\delta_{0,i}); r_l(T_{0,i}); r_u(T_{0,i}); T_1\right] \Bigg)
\end{aligned}$$

with

$$H(T_0, t) := \sum_{i=1}^{n_0^-} \mathbb{1}_{\{r_l(T_{0,i}) \le r_n(T_{0,i}, T_{0,i}+\delta_{0,i}) \le r_u(T_{0,i})\}}.$$

For the subsequent coupons, we get

$$\begin{aligned}
\nu_{j+1}(t) &= B_t \mathbb{E}\left[\frac{1}{B_{T_{j+1}}} \frac{r_n(T_j, T_{j+1}) + s_j}{D_j} H(T_j, T_{j+1}) \Big| \mathcal{F}_t\right] \\
&= P(t, T_{j+1}) \mathbb{E}_{T_{j+1}}\left[\frac{r_n(T_j, T_{j+1}) + s_j}{D_j} H(T_j, T_{j+1}) \Big| \mathcal{F}_t\right] \\
&\overset{(4.1)}{=} \left(\frac{s_j}{D_j} - \frac{1}{\delta_j D_j}\right) P(t, T_{j+1}) \sum_{i=1}^{n_j} \mathbb{E}_{T_{j+1}}\left[1\!\!1_{\{r_l(T_{j,i}) \le r_n(T_{j,i}, T_{j,i}+\delta_{j,i}) \le r_u(T_{j,i})\}} \Big| \mathcal{F}_t\right] \\
&\quad + \frac{P(t, T_{j+1})}{\delta_j D_j} \sum_{i=1}^{n_j} \mathbb{E}_{T_{j+1}}\left[\frac{1}{P(T_j, T_{j+1})} 1\!\!1_{\{r_l(T_{j,i}) \le r_n(T_{j,i}, T_{j,i}+\delta_{j,i}) \le r_u(T_{j,i})\}} \Big| \mathcal{F}_t\right] \\
&=: \nu^1_{j+1}(t) + \nu^2_{j+1}(t).
\end{aligned}$$

To evaluate $\nu^1_{j+1}(t)$ we proceed as before and get

$$\nu^1_{j+1}(t) = \left(\frac{s_j}{D_j} - \frac{1}{\delta_j D_j}\right) \sum_{i=1}^{n_j} DRD_t[r_n(T_{j,i}, T_{j,i} + \delta_{j,i}); r_l(T_{j,i}); r_u(T_{j,i}); T_{j+1}].$$

For the evaluation of $\nu^2_{j+1}(t)$ we switch from the forward measure $\mathbb{P}_{T_{j+1}}$ to the adjusted forward measure $\mathbb{P}_{T_j, T_{j+1}}$. This procedure has the advantage that we do not have to deal with the joint distribution of the two random variables $P(T_j, T_j + \delta_j)$ and $r_n(T_{j,i}, T_{j,i} + \delta_{j,i})$. Using the abstract Bayes formula together with (3.3)-(3.4) and denoting by $\mathbb{E}_{T_j, T_{j+1}}$ the expectation with respect to $\mathbb{P}_{T_j, T_{j+1}}$ yields

$$\begin{aligned}
\nu^2_{j+1}(t) &= \sum_{i=1}^{n_j} \frac{P(t, T_{j+1})}{\delta_j D_j} \mathbb{E}_{T_{j+1}}\left[\frac{1}{P(T_j, T_{j+1})} 1\!\!1_{\{r_l(T_{j,i}) \le r_n(T_{j,i}, T_{j,i}+\delta_{j,i}) \le r_u(T_{j,i})\}} \Big| \mathcal{F}_t\right] \\
&= \sum_{i=1}^{n_j} \frac{P(t, T_{j+1})}{\delta_j D_j} \frac{P(0, T_j)}{P(0, T_{j+1})} \\
&\qquad \times \mathbb{E}_{T_{j+1}}\left[\frac{F(T_j, T_j, T_{j+1})}{F(0, T_j, T_{j+1})} 1\!\!1_{\{r_l(T_{j,i}) \le r_n(T_{j,i}, T_{j,i}+\delta_{j,i}) \le r_u(T_{j,i})\}} \Big| \mathcal{F}_t\right] \\
&= \sum_{i=1}^{n_j} \frac{P(t, T_{j+1})}{\delta_j D_j} \frac{P(0, T_j)}{P(0, T_{j+1})} \frac{P(0, T_{j+1}) P(t, T_j)}{P(0, T_j) P(t, T_{j+1})} \\
&\qquad \times \mathbb{E}_{T_j, T_{j+1}}\left[1\!\!1_{\{r_l(T_{j,i}) \le r_n(T_{j,i}, T_{j,i}+\delta_{j,i}) \le r_u(T_{j,i})\}} \Big| \mathcal{F}_t\right] \\
&= \sum_{i=1}^{n_j} \frac{P(t, T_j)}{\delta_j D_j} \mathbb{E}_{T_j, T_{j+1}}\left[1\!\!1_{\{r_l(T_{j,i}) \le r_n(T_{j,i}, T_{j,i}+\delta_{j,i}) \le r_u(T_{j,i})\}} \Big| \mathcal{F}_t\right]
\end{aligned}$$

The summands on the right hand side look (except for a multiplicative constant) very similar to the time-t value of a range digital option, the only difference being that the expectation is taken under the adjusted forward measure. We can proceed in the same way as we did for digital options and use the independence of the increments of L to obtain

$$\nu^2_{j+1}(t) = \frac{P(t, T_j)}{\delta_j D_j} \sum_{i=1}^{n_j} D^{j,i}_t.$$

Here,

$$
\begin{aligned}
D_t^{j,i} &\overset{(4.1)}{=} \mathbb{E}_{T_j,T_{j+1}}\left[\mathbb{1}_{\left\{\frac{1}{\delta_{j,i} r_u(T_{j,i})+1} \le P(T_{j,i},T_{j,i}+\delta_{j,i}) \le \frac{1}{\delta_{j,i} r_l(T_{j,i})+1}\right\}} \big| \mathcal{F}_t\right] \\
&\overset{(2.4)}{=} \mathbb{E}_{T_j,T_{j+1}}\left[\mathbb{1}_{\left\{\underline{K}^{j,i} \le \frac{P(t,T_{j,i}+\delta_{j,i})}{P(t,T_{j,i})}\exp(X^{j,i}) \le \overline{K}^{j,i}\right\}} \big| \mathcal{F}_t\right] \\
&= h^{j,i}\left(\frac{P(t,T_{j,i}+\delta_{j,i})}{P(t,T_{j,i})}\right) \qquad (5.1)
\end{aligned}
$$

with $h^{j,i} : \mathbb{R}^+ \to [0,1]$ given by

$$
h^{j,i}(y) = \int \mathbb{1}_{\left\{\frac{1}{y}\underline{K}^{j,i} \le e^x \le \frac{1}{y}\overline{K}^{j,i}\right\}} \, \mathrm{d}\mathbb{P}^{X^{j,i}}_{T_j,T_{j+1}}(x) \qquad (5.2)
$$

and where

$$
\begin{aligned}
X^{j,i} &:= \int_t^{T_{j,i}} \Sigma(s, T_{j,i}, T_{j,i}+\delta_{j,i}) \, \mathrm{d}L_s, \\
\overline{K}^{j,i} &:= \frac{1}{\delta_{j,i} r_l(T_{j,i})+1} \exp \int_t^{T_{j,i}} A(s, T_{j,i}, T_{j,i}+\delta_{j,i}) \, \mathrm{d}s, \\
\underline{K}^{j,i} &:= \frac{1}{\delta_{j,i} r_u(T_{j,i})+1} \exp \int_t^{T_{j,i}} A(s, T_{j,i}, T_{j,i}+\delta_{j,i}) \, \mathrm{d}s,
\end{aligned}
$$

and $\mathbb{P}^{X^{j,i}}_{T_j,T_{j+1}}$ denotes the distribution of $X^{j,i}$ with respect to $\mathbb{P}_{T_j,T_{j+1}}$.

To improve readability, let us simplify notation and fix j and i. In what follows, we omit the sub- and superscripts j,i and write T, δ, D_t, h, X, $\overline{K}$ and $\underline{K}$ for short. Denote by M^X the moment generating function of the random variable X with respect to $\mathbb{P}_{T_j,T_{j+1}}$. Then we have the following pricing formula:

Theorem 5.2 *Suppose the distribution of X possesses a Lebesgue-density. Choose an $R > 0$ such that $M^X(-R) < \infty$. Then*

$$
\begin{aligned}
D_t &= \frac{1}{\pi}\int_0^\infty \Re\left(\left(\frac{P(t,T)}{P(t,T+\delta)}\overline{K}\right)^{R+iu} \frac{1}{R+iu} M^X(-R-iu)\right) \mathrm{d}u \\
&\quad - \frac{1}{\pi}\int_0^\infty \Re\left(\left(\frac{P(t,T)}{P(t,T+\delta)}\underline{K}\right)^{R+iu} \frac{1}{R+iu} M^X(-R-iu)\right) \mathrm{d}u
\end{aligned}
$$

with

$$
\overline{K} := \frac{1}{\delta r_l(T)+1} \exp \int_t^T A(s,T,T+\delta) \, \mathrm{d}s, \qquad (5.3)
$$

$$
\underline{K} := \frac{1}{\delta r_u(T)+1} \exp \int_t^T A(s,T,T+\delta) \, \mathrm{d}s. \qquad (5.4)
$$

Proof: Observe that

$$
h(y) = \int \mathbb{1}_{\left\{e^x \le \frac{\overline{K}}{y}\right\}} \, \mathrm{d}\mathbb{P}^X_{T_j,T_{j+1}}(x) - \int \mathbb{1}_{\left\{e^x < \frac{\underline{K}}{y}\right\}} \, \mathrm{d}\mathbb{P}^X_{T_j,T_{j+1}}(x).
$$

Applying exactly the same arguments as in the proof of Theorem 4.2 yields the claim. The only difference is that we consider the moment generating function of X with respect to an adjusted forward measure and not with respect to a forward measure. □

The next theorem gives an expression for M^X.

Theorem 5.3 *Under the assumptions of Theorem 5.2 we have for $u \in \mathbb{R}$*

$$M^X(-R-iu) = \exp \int_t^T \Big[\theta_s(g_s(-R-iu)) - \theta_s(g_s(0))\Big]\,\mathrm{d}s, \tag{5.5}$$

with $g_s(z) := z\Sigma(s,T,T+\delta) + \Sigma(s,T_j)\mathbb{1}_{\{s\leq T_j\}} + \Sigma(s,T_{j+1})\mathbb{1}_{\{T_j<s\}}$.

PROOF: Similar to the proof of Theorem 4.3. □

Once again, let us consider the special case of a multifactor Gaussian HJM model. We have $\theta(x) = \frac{\langle x,x\rangle}{2}$ for $x \in \mathbb{C}^d$ and from (5.5) we get for $z \in \mathbb{C}$

$$\begin{aligned} M^X(z) &= \exp\Bigg(\frac{z^2}{2}\int_t^T ||\Sigma(s,T,T+\delta)||^2\,\mathrm{d}s \\ &\quad + z\int_t^T \langle \Sigma(s,T,T+\delta), \Sigma(s,T_j)\mathbb{1}_{\{s\leq T_j\}} + \Sigma(s,T_{j+1})\mathbb{1}_{\{T_j<s\}}\rangle\,\mathrm{d}s\Bigg). \end{aligned}$$

Consequently, X is normally distributed under $\mathbb{P}_{T_j,T_{j+1}}$ with mean

$$m(t,T,T+\delta,T_j,T_{j+1}) := \int_t^T \langle \Sigma(s,T,T+\delta), \Sigma(s,T_j)\mathbb{1}_{\{s\leq T_j\}} + \Sigma(s,T_{j+1})\mathbb{1}_{\{T_j<s\}}\rangle\,\mathrm{d}s$$

and variance

$$g(t,T,T+\delta) := \int_t^T ||\Sigma(s,T,T+\delta)||^2\,\mathrm{d}s. \tag{5.6}$$

From (5.2) we get

$$\begin{aligned} h(y) &= \int_{\log\frac{K}{y}}^{\log\frac{\overline{K}}{y}} \mathrm{d}\mathbb{P}^X_{T_j,T_{j+1}}(x) \\ &= \mathbb{P}_{T_j,T_{j+1}}\left(\log\frac{K}{y} \leq X \leq \log\frac{\overline{K}}{y}\right) \\ &= \Phi\left(\frac{\log\frac{\overline{K}}{y} - m(t,T,T+\delta,T_j,T_{j+1})}{\sqrt{g(t,T,T+\delta)}}\right) \\ &\qquad -\Phi\left(\frac{\log\frac{K}{y} - m(t,T,T+\delta,T_j,T_{j+1})}{\sqrt{g(t,T,T+\delta)}}\right). \end{aligned}$$

Plugging in the expression for $\overline{K}$ and $\underline{K}$ from (5.3)-(5.4) and using (5.1) we end up with

$$\begin{aligned} D_t &= \Phi\left(\frac{\log \frac{P(t,T)}{P(t,T+\delta)(\delta r_l(T)+1)} + \frac{1}{2}g(t,T,T+\delta) - l(t,T,T+\delta,T_j,T_j+1)}{\sqrt{g(t,T,T+\delta)}}\right) \\ &\quad -\Phi\left(\frac{\log \frac{P(t,T)}{P(t,T+\delta)(\delta r_u(T)+1)} + \frac{1}{2}g(t,T,T+\delta) - l(t,T,T+\delta,T_j,T_j+1)}{\sqrt{g(t,T,T+\delta)}}\right), \end{aligned}$$

where

$$\begin{aligned} l(t,T,T+\delta,T_j,T_j+1) &:= \int_t^T \langle \Sigma(s,T,T+\delta), \\ &\qquad \Sigma(s,T_j)\mathbb{1}_{\{s\le T_j\}} + \Sigma(s,T_{j+1})\mathbb{1}_{\{T_j<s\}} - \Sigma(s,T)\rangle \, \mathrm{d}s. \end{aligned} \tag{5.7}$$

Putting pieces together, we obtain the following result:

Theorem 5.4 *Using the notation introduced above, the time-t price of a floating range note in the multifactor Gaussian HJM model is equal to*

$$FlRN(t) = P(t,T_N) + \sum_{j=0}^{N-1} \nu_{j+1}(t)$$

with

$$\begin{aligned} \nu_1(t) &= \frac{r_n(T_0,T_0+\delta_0)+s_0}{D_0}\Bigg(P(t,T_1)H(T_0,t) \\ &\qquad + \sum_{i=n_0^-+1}^{n_0} DRD_t\left[r_n(T_{0,i},T_{0,i}+\delta_{0,i}); r_l(T_{0,i}); r_u(T_{0,i}); T_1\right]\Bigg) \end{aligned}$$

and

$$\begin{aligned} \nu_{j+1}(t) &= \left(\frac{s_j}{D_j} - \frac{1}{\delta_j D_j}\right)\sum_{i=1}^{n_j} DRD_t[r_n(T_{j,i},T_{j,i}+\delta_{j,i}); r_l(T_{j,i}); r_u(T_{j,i}); T_{j+1}] \\ &\quad + \frac{P(t,T_j)}{\delta_j D_j}\sum_{i=1}^{n_j}\Big(\Phi(\eta_{j,i}(r_l(T_{j,i}))) - \Phi(\eta_{j,i}(r_u(T_{j,i})))\Big) \end{aligned}$$

where

$$\eta_{j,i}(r) := \frac{\log \frac{P(t,T_{j,i})}{P(t,T_{j,i}+\delta_{j,i})(\delta_{j,i}r+1)} + \frac{1}{2}g(t,T_{j,i},T_{j,i}+\delta_{j,i}) - l(t,T_{j,i},T_{j,i}+\delta_{j,i},T_j,T_j+1)}{\sqrt{g(t,T_{j,i},T_{j,i}+\delta_{j,i})}}$$

and g and l are defined as in (5.6) and (5.7).

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